Chapter 4

Linear DAEs with Time-varying Coefficients

In this section, we consider linear DAEs with *variable* or *time-dependent* coefficients. This means, for matrix-valued functions

$$E \in \mathcal{C}(\mathcal{I}, \mathbb{C}^{m,n}), \quad A \in \mathcal{C}(\mathcal{I}, \mathbb{C}^{m,n})$$

and $f \in \mathcal{C}(\mathcal{I}, \mathbb{C}^m)$, we consider the DAE

$$E(t)\dot{x}(t) = A(t)x(t) + f(t) \tag{4.1}$$

with, possibly, an initial condition

$$x(t_0) = x_0 \in \mathbb{C}^n. \tag{4.2}$$

> The same general solution concept applies. Basically x should be differentiable, fulfill the DAE, and, if stated, the initial condition too.

In the constant coefficient case, regularity played a decisive role for the existence and uniqueness of solutions; see, e.g. Section 3.4. Thus it seems natural to extend this concept to the time-varying case, e.g., through requiring that $(E(t), A(t))$ is a regular matrix pair independent of t. However, the following two examples show that this will not work *out of the box*.

Example 4.1. Let E, A be given as

$$E(t) = \begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then

$$\det(\lambda E(t) - A(t)) = (1 - \lambda t)(1 + \lambda t) + \lambda^2 t^2 \equiv 1,$$

for all $t \in \mathcal{I}$. Still, for every $c \in \mathcal{C}^1(\mathcal{I}, \mathbb{C})$ with $c(t_0) = 0$, the function

$$x \colon t \mapsto c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$$

solves the *homogeneous* initial value problem (4.1)–(4.2).

> This was an example where the pair (E, A) is regular uniformly with respect to t but still allows for infinitely many solutions to the associated DAE. **X**: What about the initial value? Why it won't help to make the solution unique?
>
> Next we see the contrary – a matrix pair that is singular for any t but defines a unique solution.

Example 4.2. For

$$E(t) = \begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}, \quad A(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

one has

$$\det(\lambda E(t) - A(t)) = 0$$

for all $t \in \mathcal{I}$. Still, if $x = (x_1, x_2)$ denotes the solution, from the first line of the DAE

$$\begin{aligned} 0 &= -x_1(t) + t x_2(t) + f_1(t) \\ \dot{x}_1 - t\dot{x}_2(t) &= \qquad\qquad\qquad\quad f_2(t) \end{aligned}$$

one can calculate directly that

$$\dot{x}_1(t) = t\dot{x}_2(t) + x_2 + \dot{f}_1(t)$$

or that

$$\dot{x}_1(t) - t\dot{x}_2(t) = x_2 + \dot{f}_1(t)$$

so that the second line becomes

$$x_2(t) + \dot{f}_1(t) = f_2(t)$$

which uniquely defines

$$x_2(t) = -\dot{f}_1(t) + f_2(t)$$

and also

$$x_1(t) = -t(\dot{f}_1(t) + f_2(t)) + f_1(t).$$

> For both examples one can then simply choose $x(t_0)$ in accordance with the right hand side to argue about whether and how a solution exists.

Recall that for the *constant coefficient* case, we were using invertible scaling and state transformation matrices P and Q for the equivalence transformations

$$E\dot{x}(t) = Ax(t) + f(t) \quad \sim \quad \tilde{E}\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{f}(t)$$

with

$$x = Q\tilde{x}, \quad \tilde{E} = PEQ, \quad \tilde{A} = PAQ, \quad \tilde{f} = Pf.$$

For the time-varying case, we will use time-varying transformations and require that they are invertible at every point t in time.

Definition 4.1. Two pairs (E_i, A_i), E_i, $A_i \in \mathcal{C}(\mathcal{I}, \mathbb{C}^{m,n})$, $i = 1, 2$, of matrix functions are called *(globally) equivalent*, if there exist pointwise nonsingular matrix functions $P \in \mathcal{C}(\mathcal{I}, \mathbb{C}^{m,m})$ and $Q \in \mathcal{C}^1(\mathcal{I}, \mathbb{C}^{n,n})$ such that

$$E_2 = PE_1Q, \quad A_2 = PA_1Q - PE_1\dot{Q} \tag{4.3}$$

for all $t \in \mathcal{I}$. Again, we write $(E_1, A_1) \sim (E_2, A_2)$.

The need of Q being differentiable and the appearance of $E_1\dot{Q}$ in the definition of A_2 comes from the relation

$$E\dot{x}(t) = E\frac{d}{dt}(Q\tilde{x})(t) = E(Q(t)\dot{\tilde{x}}(t) + \dot{Q}(t)\tilde{x}(t))$$

for the transformed state $\tilde{x}$ with the actual state x.

Lemma 4.1. *The relation on pairs of matrix functions as defined in Definition 4.1 is an equivalence relation.*

Proof. Exercise! □

Next we will define *local* equivalence of matrix pairs.

Definition 4.2. Two pairs (E_i, A_i), E_i, $A_i \in \mathbb{C}^{m,n}$, $i = 1, 2$, of matrices are called *locally equivalent*, if there exist pointwise nonsingular matrices $P \in \mathbb{C}^{m,m})$ and $Q \in \mathbb{C}^{n,n}$ such that as well as matrix $R \in \mathbb{C}^{n,n}$ such that

$$E_2 = PE_1Q, \quad A_2 = PA_1Q - PE_1R. \tag{4.4}$$

Again, we write $(E_1, A_1) \sim (E_2, A_2)$ and differentiate by context.

Lemma 4.2. *The local equivalence as defined in Definition 4.2 is an equivalence relation on pairs of matrices.*

Proof. Exercise! □

We state a few observations:

- Global equivalence implies local equivalence at all points of time t.
- Vice versa, pointwise local equivalence, e.g. at some time instances t_i with suitable matrices P_i, Q_i, R_i, can be interpolated to a continuous matrix function P and a differentiable matrix function Q by *Hermite interpolation*, i.e. via

$$P(t_i) = P_i, \quad Q(t_i) = Q_i, \quad \dot{Q}(t_i) = R_i.$$

- Local equivalence is more powerful than the simple equivalence of matrix pairs (cp. Definition 3.1) for which $R = 0$. This means we can expect more structure in a normal form.

4.1 A Local Canonical Form

$M = [v_1, \dots, v_k]$

$\sim M = \operatorname{span}\{v_1 .. v_k\}$

For easier explanations, we introduce the slightly incorrect wording that a *matrix* M *spans* a vector space V to express that the V is the span of the columns of V. Similarly, we will say that M *is a basis of* V, if the columns of M form a basis for V.

Some more notation:

$T \in \mathbb{R}^{n,k}$

$\left[v_1 \dots, v_k \,\middle|\, v_{k+1} \dots v_n \right]$

T'

$[T | T']$ inv.

Notation	Explanation
$V^H \in \mathbb{C}^{n,m}$	the *conjugate transpose* or *Hermitian transpose* of a matrix $V \in \mathbb{C}^{m,n}$
$T' \in \mathbb{C}^{n,n-k}$	The *complementary space* as a matrix. If $T \in \mathbb{C}^{n,k}$ is a basis of V, then T' contains a basis of V' so that $V \oplus V' = \mathbb{C}^n$. In particular, the matrix $[T \quad T']$ is square and invertible.

$E\dot{x} = Ax + f$

Theorem 4.1. *Let* $E, A \in \mathbb{C}^{m,n}$ *and let*

$$T,\ Z,\ T',\ V \tag{4.5}$$

be

$ET = 0$

$[T\ T']$

Matrix	*as the basis of*
T	kernel E
Z	corange E = kernel E^H
T'	cokernel E = range E^H
V	corange$(Z^H AT)$

then the quantities

$$r,\ a,\ s,\ d,\ u,\ v \tag{4.6}$$

defined as

Quantity	*Definition*	*Name*
r	$\operatorname{rank} E$	rank
a	$\operatorname{rank}(Z^H AT)$	algebraic part
s	$\operatorname{rank}(V^H Z^H AT')$	strangeness
d	$r - s$	differential part
u	$n - r - a$	undetermined variables
v	$m - r - a - s$	vanishing equations

Schiefheit

are invariant under local equivalence transformations and (E, A) *is locally equivalent to the canonical form*

$$(E,A) \sim \left(\begin{bmatrix} \boxed{I_s} & 0 & 0 & 0 \\ 0 & \boxed{I_d} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{I_a} & 0 \\ \boxed{I_s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right), = (PEQ, PAQ - PER) \tag{4.7}$$

$\dot{x}_1 = f_1$

$0 = x_1 + f_4$

where all diagonal blocks are square, except maybe the last one.

Proof. To be provided. Until then, see Theorem 3.7 in Kunkel/Mehrmann. $\square$

Some remarks on the spaces and how the names are derived for the case $E\dot{x} = Ax + f$ with constant coefficients. The ideas are readily transferred to the case with time-varying coefficients.

$E(t)\dot{x}(t)$

$E(t)T(t) = 0$

Let

$$x(t) = Ty(t) + T'y'(t),$$

where y denotes the components of x that evolve in the range of T and y' the respective complement. (Since $[T|T']$ is a basis of $\mathbb{C}^n$, there exist such y and y' that uniquely define x and vice versa). With T spanning $\ker E$ we find that

$$x(t) = [T\ T']\underbrace{[T\ T']^{-1}x(t)}_{\tilde{y}(t)} = [T\ T']\tilde{y}(t) = [T \vdots T']\begin{bmatrix} y \\ y' \end{bmatrix}(t)$$

→ y wird nicht differenziert

$$E\dot{x}(t) = E\cancel{T}\dot{y}(t) + ET'\dot{y}'(t) = ET'\dot{y}'(t)$$

so that the DAE basically reads

y' wird differenziert

$$ET'\dot{y}'(t) = ATy(t) + AT'y'(t) + f,$$

i.e. the components of x defined through y are, effectively, not differentiated. With Z containing exactly those v, for which $v^H E = 0$, it follows that

$$Z^H \cancel{E}T'\dot{y}'(t) = 0 = \boxed{Z^H ATy(t)} + Z^H AT'y'(t) + Z^H f,$$

$$E\dot{x} = \cancel{ET\dot{y}(t)} + ET'\dot{y}'(t)$$

$$\begin{bmatrix} I & \\ & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}(t) = \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix}$$

or

α

$$Z^H AT\boxed{y(t)} = -Z^H AT' y'(t) - Z^H f,$$

so that rank $Z^H AT$ indeed describes the number of purely algebraic equations and variables in the sense that it defines parts of y (which is never going to be differentiated) in terms of algebraic relations (no time derivatives are involved).

With the same arguments and with $V = \operatorname{corange} Z^H AT$, it follows that

s

$$V^H Z^H AT' y'(t) = -V^H \cancel{Z^H AT} y(t) - V^H Z^H f = -V^H Z^H f,$$

is the part of $E\dot{x} = Ax + f$ in which those components y' that are also differentiated are algebraically equated to a right-hand side. This is the *strangeness* (rather in the sense of *skewness*) of DAEs that variables can be both differential and algebraic. Accordingly, rank $V^H Z^H AT'$ describes the size of the skewness component.

Finally, those variables that are neither *strange* nor purely algebraic, i.e. those that are differentiated but not defined algebraically, are the *differential* variables. There is no direct characterization of them, but one can calculate their number as $r-s$, which means number of differentiated minus number of *strange* variables.

> **Outlook**: If there is no strangeness, the DAE is called strangeness-free. Strangeness can be eliminated through iterated differentiation and substitution. The needed number of such iterations (that is independent of the the size s of the *strange* block here) will define the strangeness index.

Example 4.3. With a basic state transformation

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} x_3 - x_2 \\ x_2 - x_1 \\ x_3 \end{bmatrix},$$

one finds for the coefficients of Example 1.2 that:

$$(E, A) \backsim \left(\begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{R} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

We compute the subspaces as defined in (4.5):

Matrix	as the basis of/computed as
$T = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$	kernel $\begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix}$

Matrix	as the basis of/computed as
$Z = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$	$\operatorname{corange} \begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix} = \operatorname{kernel} \begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix}^H$
$T' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\operatorname{cokernel} \begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix} = \operatorname{range} \begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix}^H$
$Z^H AT = I_2$	$\begin{bmatrix} 0 \\ I_2 \end{bmatrix}^H \begin{bmatrix} 0 & \frac{1}{R} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$
$V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\operatorname{corange}(Z^H AT) = \operatorname{kernel} I_2^H$
$V^H Z^H AT' = [0]$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}^H \begin{bmatrix} 0 \\ I_2 \end{bmatrix}^H \begin{bmatrix} 0 & \frac{1}{R} & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and derive the quantities as defined in (4.6):

Name	Value	Derived from
rank	$r = 1$	$\operatorname{rank} E = \operatorname{rank} \begin{bmatrix} C & 0 \\ 0 & 0_2 \end{bmatrix}$
algebraic part	$a = 2$	$\operatorname{rank} Z^H AT = \operatorname{rank} I_2$
strangeness	$s = 0$	$\operatorname{rank} V^H Z^H AT' = \operatorname{rank} [0]$
differential part	$d = 1$	$d = r - s = 1 - 0$
undetermined variables	$u = 0$	$u = n - r - a = 3 - 2 - 1$
vanishing equations	$v = 0$	$v = m - r - a - s = 3 - 2 - 1 - 0$

Example 4.4. With more involved scalings and state transforms, one finds for the coefficients of the linearized and spatially discretized Navier-Stokes equations (see Exercise I) that:

$$(\mathcal{E}, \mathcal{A}) = \left(\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B^H \\ B & 0 \end{bmatrix} \right) \backsim \left(\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & I_{n_1} \\ A_{21} & A_{22} & 0 \\ I_{n_1} & 0 & 0 \end{bmatrix} \right).$$

We compute the subspaces as defined in (4.5):

Matrix	as the basis of/computed as
$T = \begin{bmatrix} 0 \\ 0 \\ I_{n_1} \end{bmatrix}$	kernel $\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$Z = \begin{bmatrix} 0 \\ 0 \\ I_{n_1} \end{bmatrix}$	corange $\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$\underline{T'} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix}$	cokernel $\begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$Z^H AT = 0_{n_1}$	$\begin{bmatrix} 0 \\ 0 \\ I_{n_1} \end{bmatrix}^H \begin{bmatrix} A_{11} & A_{12} & I_{n_1} \\ A_{21} & A_{22} & 0 \\ I_{n_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ I_{n_1} \end{bmatrix}$
$V = I_{n_1}$	$\operatorname{corange}(Z^H AT) = \operatorname{kernel} 0_{n_1}^H$
$Z^H AT' =$ $\begin{bmatrix} I_{n_1} & 0_{n_1 \times n_2} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ I_{n_1} \end{bmatrix}^H \begin{bmatrix} A_{11} & A_{12} & I_{n_1} \\ A_{21} & A_{22} & 0 \\ I_{n_1} & 0 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} & 0 \\ 0 & I_{n_2} \\ 0 & 0 \end{bmatrix}$

and derive the quantities as defined in (4.6):

Name	Value	Derived from
rank	$r = n_1 + n_2$	$\operatorname{rank} E = \operatorname{rank} \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$
algebraic part	$a = 0$	$\operatorname{rank} Z^H AT = \operatorname{rank} 0_{n_1}$
strangeness	$s = \boxed{n_1}$	$\operatorname{rank} V^H Z^H AT' = \operatorname{rank} \begin{bmatrix} I_{n_1} & 0_{n_1 \times n_2} \end{bmatrix}$
differential part	$d = n_2$	$d = r - s = (n_1 + n_2) - n_1$
undetermined variables	$u = \boxed{n_1}$	$u = n - r - a =$ $(n_1 + n_2 + n_1) - (n_1 + n_2) - 0$
vanishing equations	$v = 0$	$v = m - r - a - s =$ $(n_1 + n_2 + n_1) - (n_1 + n_2) - n_1$

4.2 Don't read any further

Theorem 4.2 (see Kunkel/Mehrmann, Thm. 3.9)**.** *Let* $E \in \mathcal{C}^l(I, \mathbb{C}^{m,n})$ *with* $\operatorname{rank} E(t) = r$ *for all* $t \in I$*. Then there exist smooth and pointwise unitary (and, thus, nonsingular) matrix functions* U *and* V*, such that*

$$U^H EV = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}$$