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Differential Algebraic Equations

Exercise Sheet I – Introductory Considerations and Basic Notions

A Multi-body systems
Many multibody system can be modelled

$$\begin{aligned} M\dot{x} &= Mv && \text{(1a)}\\ M\dot{v} &= Ax + Kv - G(x)^T\lambda + f, && \text{(1b)}\\ 0 &= g(x), && \text{(1c)} \end{aligned}$$

with $x(t) \in \mathbb{R}^n$, $\lambda(t) \in \mathbb{R}^m$, $M \in \mathbb{R}^{n,n}$ symmetric strictly positive definite, $A \in \mathbb{R}^{n,n}$, $g\colon \mathbb{R}^n \to \mathbb{R}^m$, and

$$G(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \dots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \dots & \frac{\partial g_m}{\partial x_n}(x) \end{bmatrix},$$

is the Jacobian of g at x, with $m < n$ and $G(x)$ having full rank for any x with $g(x) = 0$.

1. Show that the pendulum (lecture: Exa. 1.1) can be brought into the form (1).
2. Find, write down, and explain another multibody system that can be modelled in the form of (1).

B Separation of differential and algebraic parts
Under certain regularity assumptions, a general DAE

$$\mathcal{F}(t, x, \dot{x}) = 0$$

can locally be brought into the form

$$\begin{aligned} \dot{x}_1 &= \mathcal{L}(t, x_1, x_2), && \text{(2a)}\\ x_2 &= \mathcal{R}(t, x_1), && \text{(2b)} \end{aligned}$$

by means of differentiation, elimination, and the splitting $x = [x_1, x_2]$.

1. Bring the *circuit* example (lecture: Exa. 1.2) into the form (2) with $\mathcal{L}$ and $\mathcal{R}$ defined explicitly. How many differentiations did you need? What were the necessary regularity conditions?
2. Bring the *pendulum* example (lecture: Exa. 1.1) into the form (2). Here, $\mathcal{L}$ and $\mathcal{R}$ may be defined implicitly. How many differentiations did you need? What were the necessary regularity conditions?
3. How can one express consistency conditions for an intial value x_0 by means of formulation (2).

C Modelling the pendulum anew
The pendulum can also be modelled as a pure ODE, e.g., by means of certain *generalized coordinates*. Present such a model and discuss advantages of the different formulations in view of a general multibody system or *Automatic Modelling*.

D Spatially discretized linearized Navier-Stokes equations
In simulations of flows, equations of the form

$$M\dot{v} = Av - B^T p, \quad v(0) = v_0 \in \mathbb{R}^n \tag{3a}$$

$$0 = Bv - g \tag{3b}$$

with $v(t) \in \mathbb{R}^n$, $p(t) \in \mathbb{R}^m$, $g(t) \in \mathbb{R}^m$, and A, $M \in \mathbb{R}^{n,n}$, and $B \in \mathbb{R}^{m,n}$, such that M is invertible as is $BM^{-1}B^T$. Equation (2) typically represents a spatially discretized and linearized Navier-Stokes equation.

1. Reformulate (3) in the form of (2). How many differentiations did you need? What were the necessary regularity conditions?
2. Write down an ODE initial value problem for v and express its solution explicitly. (Hint: *Variation of Constants*)
3. Prove that any v that solves the initial value problem (3) also solves the ODE from **D**.2. What about the converse direction?

D Equivalence and Regularity of Matrix Pairs
Show that

1. the relation of *strong equivalence* as defined in Definition 3.1 is an equivalence relation,
2. regularity of a matrix pair (Definition 3.5) is invariant under *strong equivalence*.

E Singular Blocks in the Kronecker Canonical Form
Explain how the singular blocks

$$\lambda \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

fit into the *Kronecker Canonical Form* (Theorem 3.3).