MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Robust control for compensation of linearization and discretization errors in stabilization of incompressible flows

Peter Benner, Steffen W. R. Werner, Jan Heiland

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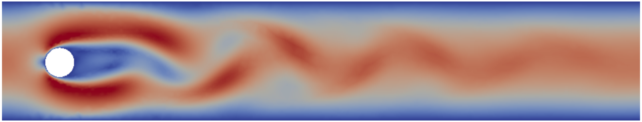
Universität Passau

Outline

1. Introduction
2. Uncertain Linearization Points are Coprime Factor Uncertainties
3. Robust Controller Design
4. Application to Incompressible Flows
5. Numerical Example
6. Conclusions

Introduction
Flow Control Problem I

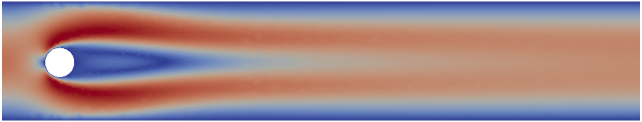


Feedback Control

Problem: The steady state is unstable: any perturbation – no matter how small – will trigger a transition into a periodic regime.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- short evaluation time,
- system uncertainties.



Idea: Linearization-based feedback control for stabilization of the steady state.

[Raymond'05/'06, PB&JH'15, Breiten&Kunisch'14]

$$\begin{aligned} \dot{v} + (v \cdot \nabla) v - \nu \Delta v + \nabla p &= Bu, \\ \nabla \cdot v &= 0 \end{aligned}$$

Linearization & Semi-Discretization →

$$\begin{aligned} \dot{v} - Av - J^{\mathsf{T}} p &= Bu, \\ Jv &= 0 \end{aligned}$$

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Linearization & Semi-Discretization →

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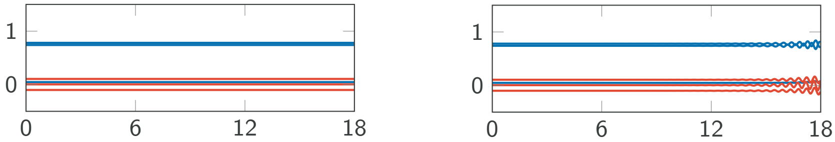
Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.

LQG-feedback   corrupted LQG-feedback



Introduction

In fact: [IEEE Transaction on Automatic Control ('78)]:

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

***Abstract*—There are none.**

Good news: Uncertainties that come from

- [Curtain'03]: Galerkin approximations of evolution systems,
- [PB&JH'17]: stable mixed-FEM approximation of the flow equations,
- [this talk, PB&JH'16]: errors in the linearization point,

can be qualified as a coprime factor perturbation of the associated transfer function.

Even better news:

- [this talk]: We can employ robust observer/controller design.

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Uncertain Linearization Points are Coprime Factor Uncertainties

Transfer functions

Mapping of inputs (controls) to outputs (measurements) in frequency domain, i.e., after Laplace transform of the system.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\mathcal{L}(s)} \quad \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) \end{aligned}$$

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1. A *nominal* system has the transfer function
$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

2. But uncertainty in the operator gives another transfer function

$$G_\Delta(s) = C(sI - A - A_\Delta)^{-1}B \in \mathbb{C}^{q,r}.$$

Coprime Factorization

Given a transfer function $G(s)$ of a linear system,

$$G(s) = M^{-1}(s)N(s)$$

is a **(left) coprime factorization** if there exist $X(s)$, $Y(s)$ such that the Bezout identity

$$M(s)X(s) + N(s)Y(s) = I$$

holds. Here, N, M, X, Y are all rational matrix functions with all poles in the open left half of the complex plane, i.e., they all represent stable linear systems.

Fact: N, M are coprime $\iff$ N, M have no common zeros in the right half plane.

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Coprime Factor Perturbation

$$G_\Delta(s) = [N(s) + N_\Delta(s)]\,[M(s) + M_\Delta(s)]^{-1}(s) \approx G(s) = N(s)M^{-1}(s),$$

where $N + N_\Delta$, $M + M_\Delta$ are stable.

Uncertain Linearization Points are Coprime Factor Uncertainties

Consider a state linear system (A, B, C) with

- $A\colon \mathcal{D}(A) \subset Z \to Z$ a generator of a C_0-semigroup
- $B\colon U \to Z$ bounded and $C\colon Z \to Y$ bounded
- U, Y, Z Hilbert spaces, U, Y finite dimensional

and

$$(A_\Delta, B, C) \sim G_\Delta \approx G \sim (A, B, C)$$

with a certain difference in the dynamics which is caused, say, by an inexact linearization.

Theorem (PB&JH '16)

If (A_Δ, B, C) and (A, B, C) are jointly stabilizable (or detectable), i.e., there exists a state feedback K (or L) that stabilizes (or makes detectable) both (A_Δ, B, C) (or (A, B, C))[1], then G differs from G_Δ through a coprime factor perturbation

$$\Delta = \begin{bmatrix} M_\Delta & N_\Delta \end{bmatrix}$$

with $\|\Delta\|_\infty \to 0$ as $A \to A_\Delta$ in the operator norm.

[1] That is, $A + BK$ and $A_\Delta + BK$ or $A + LC$ and $A_\Delta + LC$ are all stable.

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Robust controllers for coprime factor uncertainty

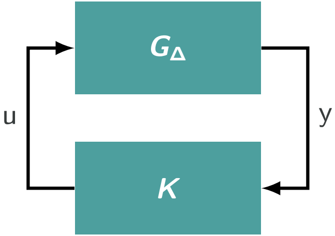
An admissable controller K stabilizes

$$G_\Delta = (M + M_\Delta)^{-1}(N + N_\Delta)$$

for all $\|\Delta\|_\infty = \|[M_\Delta \ N_\Delta]\|_\infty < \epsilon$,

if and only if[2]

- K stabilizes $G = M^{-1}N$ and
- $\| \begin{bmatrix} K \\ I \end{bmatrix} (I - GK)^{-1} M^{-1} \|_\infty \leq \epsilon^{-1}$.



Design of robust controllers

In finite dimensions, the Riccati based $\mathcal{H}_\infty$-controller with parameter γ is robustly stabilizing with $\epsilon = \gamma^{-1}$; see Cor. 3.9 in [McFarlane&Glover'90].

[2]See, e.g. [McFarlane&Glover'90] for the finite dimensional case and [Curtain&Zwart'95] for the infinite dimensional case

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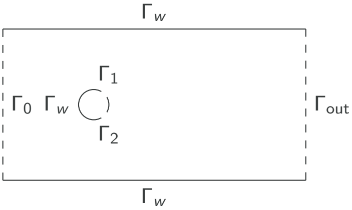
Application to Incompressible Flows

We consider



$$\dot{V} + (V \cdot \nabla)V + \nabla P - \nu \Delta V = 0,$$
$$\operatorname{div} V = 0, \quad \text{in } \Omega,$$

$$\nu \frac{\partial V}{\partial n} - nP = 0 \text{ on } \Gamma_{out},$$
$$V = 0 \text{ on } \Gamma_w,$$
$$V = ng_0 \cdot \alpha \text{ on } \Gamma_0,$$
$$V = ng_1 \cdot u_1 \text{ on } \Gamma_1,$$
$$V = ng_2 \cdot u_2 \text{ on } \Gamma_2,$$

where

- V . . . velocity,
- P . . . pressure,
- ν . . . diffusion parameter,
- g_0, g_1, g_2 . . . spatial shape functions,
- u_1, u_2 . . . scalar input functions,
- α . . . magnitude of the inflow velocity,
- n . . . normal vector at the boundaries.

Application to Incompressible Flows

To design a controller, we proceed as follows

1. We relax the Dirichlet control $V\big|_{\Gamma_1} = g_1 u - \varepsilon(\nu \frac{\partial V}{\partial n} - Pn)$
2. Let v_α be the steady state solution for zero inputs, and let $v_\delta(t) = V(t) - v_\alpha$ the deviation.
3. We consider the linearization

$$\dot{v}_\delta + (v_\delta \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)v_\delta + \nabla p_\delta - \nu\Delta v_\delta = 0$$

that is a valid approximation as long as v_δ is small.

$$\dot v_\delta + (v_\delta \cdot \nabla) v_\alpha + (v_\alpha \cdot \nabla) v_\delta + \nabla p_\delta - \nu \Delta v_\delta = 0$$
$$\operatorname{div} v_\delta = 0$$

Then, with

$$\mathcal{H}_{div} := \{v \in L^2(\Omega) : \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma_w \cap \Gamma_{\text{out}}\}$$

as the state space, the (orthogonal) *Leray*-projector

$$\Pi \in \mathcal{L}(L^2(\Omega))\colon L^2(\Omega) \mapsto \mathcal{H}_{div},$$

and $x := \Pi v_\delta$ the model reads[3]

$$\dot x = A_\alpha x + \Pi B u \quad \text{in } \mathcal{H}_{div},$$
$$y = Cx$$

where

- $A_\alpha \colon \mathcal{D}(A_\alpha) \subset \mathcal{H}_{div} \to \mathcal{H}_{div}$ is the *Oseen* operator
- $\Pi B \colon \mathbb{R}^2 \to \mathcal{H}_{div}$ is the input operator
- $C \colon \mathcal{H}_{div} \to \mathbb{R}^q$ is the output operator

[3] The pressure p_δ is gone, since Π maps along the orthogonal complement of the gradient

Application to Incompressible Flows

✓ The linearized model is a standard (A, B, C) system

- we know: A_α is the generator of a C_0-semi group [Raymond'06]
- we choose: C to be bounded
- we show below: ΠB is bounded.

➙ The theory for robust stabilization of linearization errors applies.

As for the input on Γ_1 (and similarly for Γ_2):

- The input operator is defined via

$$\langle B_1 u, w\rangle = -\frac{1}{\varepsilon}\int_{\Gamma_1} ng_1 w \, \mathrm{ds} \cdot u, \quad w \in \mathcal{H}_{div},$$

  - as it comes from the integration by parts of $\langle -\nu\Delta V + \nabla P, w\rangle$
  - and the definition of the Robin conditions.

✓ This operator is bounded as a map $B_1 \colon \mathbb{R} \to \mathcal{H}_{div}$, since:
  - $\sup_{u\in\mathbb{R}, |u|=1} \|B_1 u\|_{X^*} < \infty$ if $\sup_{w\in X, \|w\|=1} |\frac{1}{\varepsilon}\int_{\Gamma_1} ng_1 w \, \mathrm{ds}| < \infty$,
  - the *trace operator* $w \mapsto nw\big|_{\Gamma_1}$ is bounded for $X = \mathcal{H}_{div}$,
  - and since $\mathcal{H}_{div}$ is a closed subspace of $L^2(\Omega)$ so that $\mathcal{H}_{div} \simeq (\mathcal{H}_{div})^*$,

→ provided that the shape function g_1 is sufficiently smooth.

- Interestingly, $B \colon U \to L^2(\Omega)$ is not bounded, but $\Pi B \colon U \to L^2(\Omega)$ is.

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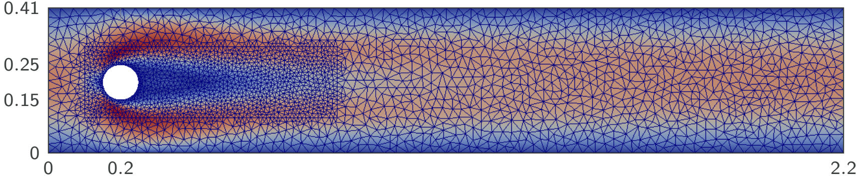


Fig.: 2D cylinder wake, discretized by Taylor-Hood (P_2/P_1) finite elements.

- Navier-Stokes equation
- Reynolds 90
- 9,843 velocity nodes
- distributed observations:
  - 3 sensors in the wake
  - measuring both v components

Target

Stabilize the steady-state and compensate perturbations to suppress vertex shedding.

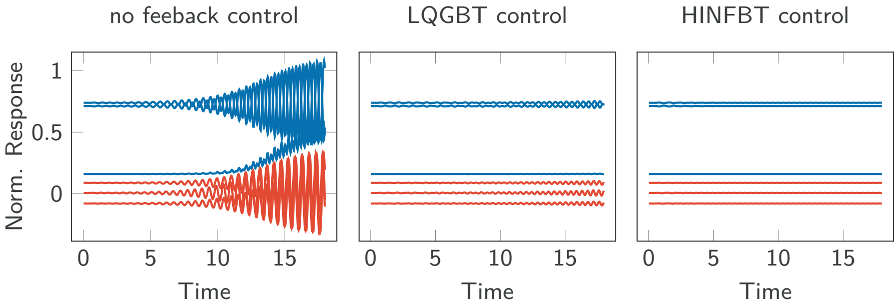
- boundary control:
  - 2 outlets at the cylinder periphery
  - control by injection and suction

Numerical Example

ℓ	$\frac{\|v_\infty - v_\ell\|}{\|v_\infty\|}$	$\|\Delta_\ell\|_{\mathcal{H}_\infty}$	γ_ℓ^{-1}
3	0.094	2.323	0.103
5	0.030	0.579	0.204
6	0.018	0.168	0.233
7	0.011	0.226	0.237
8	0.006	0.123	0.240
10	0.002	0.028	0.242

Test setup:

- v_∞ the (exact) steady state
- $v_\ell \approx v_\infty$ computed by ℓ *Picard* steps starting from the Stokes-solution
- $A := A(v_\infty)$ the exact linearization
- $A + A_\Delta := A_\ell$ the inexact linearization about v_ℓ
- Δ_ℓ the difference in the coprime factorizations
- see [PB&JH&SW'19] for how to compute the norms.



- error in linearization: 8%
- reduced-order controller dimension: 7
- trigger of instabilities by input disturbance on time interval $[0, 1]$:

$$u_\delta(t) = \begin{bmatrix} 0.01\sin(2t\pi) \\ -0.01\sin(2t\pi) \end{bmatrix}$$

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Summary

Robust controller

- that compensate linearization errors
- can be analysed via coprime factorizations and
- can be designed with Riccati-based $\mathcal{H}_\infty$-theory.

The general ∞-dimensional theory

- applies to control of incompressible flows
- if Dirichlet control is relaxed as Robin control.

In finite dimensional simulations, we can

- compute the errors in the factorization
- and provide controller with guaranteed robustness.

Outlook

- Quantify the error in the factorizations.
- Incorporate the discretization error in the controller design.

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