MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Linearization errors as smooth perturbations of coprime factors in linearized Navier-Stokes equations

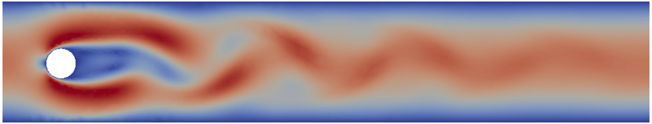
Jan Heiland

2021, September 30

M19 – Dynamics, stability and control in infinite dimensions,
@DMV-ÖMV 2021, Passau

Outline

Introduction
Flow Control Problem I

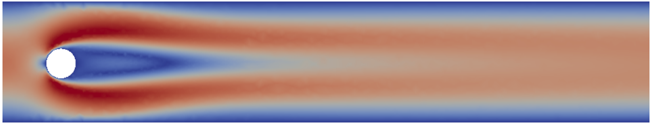


Feedback Control

Problem: The steady state is unstable: any perturbation – no matter how small – will trigger a transition into a periodic regime.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- system uncertainties.



Idea: Linearization-based feedback control for stabilization of the steady state.
[Raymond'05/'06, Benner&JH'15, Breiten&Kunisch'14]

$$\dot v + (v \cdot \nabla)v - \nu \Delta v + \nabla p = Bu, \qquad \nabla \cdot v = 0$$

Linearization & Semi-Discretization →

$$\dot v - Av - J^{\mathsf T} p = Bu, \qquad Jv = 0$$

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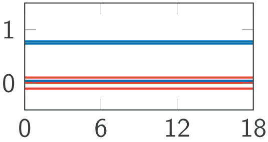
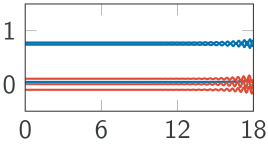
Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.

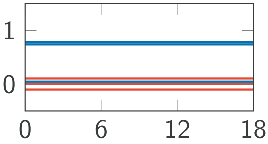
LQG-feedback



corrupted LQG-feedback



corrupted state-feedback



Introduction

In fact: [IEEE Transaction on Automatic Control ('78)]:

> **Guaranteed Margins for LQG Regulators**
>
> JOHN C. DOYLE
>
> *Abstract*—There are none.

Good news: Uncertainties that come from

- [Curtain'03]: Galerkin approximations of evolution systems,
- [Benner&JH'17]: stable mixed-FEM approximation of the flow equations,
- [Benner&JH'16]: errors in the linearization point,

can be qualified as a coprime factor perturbation of the associated transfer function.

Moreover,

- [this talk, JH'21]: the coprime factor perturbation depends smoothly on the linearization error.

Outline

1. Introduction
2. Uncertain Linearization Points are Coprime Factor Uncertainties
3. Oseen Equations as Linear System
4. Conclusions

Transfer functions

Mapping of inputs (controls) to outputs (measurements) in frequency domain, i.e., after Laplace transform of the system.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\mathcal{L}(s)} \quad \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) \end{aligned}$$

Uncertain Linearization Points are Coprime Factor Uncertainties

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

2. But uncertainty in the operator gives another transfer function

$$G_\Delta(s) = C(sI - A - \delta_A)^{-1}B \in \mathbb{C}^{q,r}.$$

Uncertain Linearization Points are Coprime Factor Uncertainties

Coprime Factorization

Given a transfer function $G(s)$ of a linear system,

$$G(s) = M^{-1}(s)N(s)$$

is a **(left) coprime factorization** if there exist $X(s), Y(s)$ such that the Bezout identity

$$M(s)X(s) + N(s)Y(s) = I$$

holds. Here, N, M, X, Y are all rational matrix functions with all poles in the open left half of the complex plane, i.e., they all represent stable linear systems.

Fact: N, M are coprime $\iff$ N, M have no common zeros in the right half plane.

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Coprime Factor Perturbation

$$G_\Delta(s) = [N(s) + \delta_N(s)]\,[M(s) + \delta_M(s)]^{-1}(s) \approx G(s) = N(s)M^{-1}(s),$$

where $N + \delta_N, M + \delta_N$ are stable.

Uncertain Linearization Points are Coprime Factor Uncertainties

Next we will show that

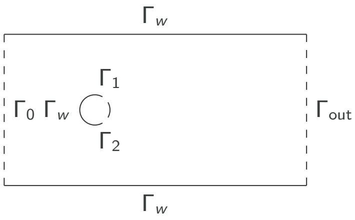
- Inexact linearizations of incompressible Navier-Stokes equations
- can be qualified as a coprime factor uncertainty
- that smoothly depends on the linearization error.

So that the standard H_∞-theory for robust controller design applies.

Outline

Oseen Equations as Linear System

We consider



$$\dot V + (V\cdot\nabla)V + \nabla P - \nu\Delta V = 0,$$
$$\operatorname{div} V = 0, \quad \text{in } \Omega,$$

$$\nu\frac{\partial V}{\partial n} - nP = 0 \text{ on } \Gamma_{out},$$
$$V = 0 \text{ on } \Gamma_w,$$
$$V = ng_0\cdot\alpha \text{ on } \Gamma_0,$$
$$V = ng_1\cdot u_1 \text{ on } \Gamma_1,$$
$$V = ng_2\cdot u_2 \text{ on } \Gamma_2,$$

where

- V ...velocity,
- P ...pressure,
- ν ...diffusion parameter,
- $g_0,\ g_1,\ g_2$ ...spatial shape functions,
- $u_1,\ u_2$ ...scalar input functions,
- α ...magnitude of the inflow velocity,
- n ...normal vector at the boundaries.

Oseen Equations as Linear System

A linearized I/O model is obtained as follows:

1. We relax the Dirichlet control $V\big|_{\Gamma_1} = ng_1 u - \varepsilon(\nu \frac{\partial V}{\partial n} - Pn)$
2. Let v_α be the steady state solution for zero inputs, and let $v_\delta(t) = V(t) - v_\alpha$ the deviation.
3. We consider the linearization

$$\dot{v}_\delta + (v_\delta \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)v_\delta + \nabla p_\delta - \nu\Delta v_\delta = 0$$

that is a valid approximation as long as v_δ is small.

Oseen Equations as Linear System

Then, with

$$\mathcal{H}_{div} := \{v \in L^2(\Omega) : \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma_w \cap \Gamma_{\text{out}}\}$$

as the state space, the (orthogonal) *Leray*-projector

$$\Pi \in \mathcal{L}(L^2(\Omega)) \colon L^2(\Omega) \mapsto \mathcal{H}_{div},$$

and $x := \Pi v_\delta$ the model reads[1]

$$\begin{aligned} \dot{x} &= A_\alpha x + \Pi B u \quad \text{in } \mathcal{H}_{div}, \\ y &= Cx \end{aligned}$$

where

- $A_\alpha \colon \mathcal{D}(A_\alpha) \subset \mathcal{H}_{div} \to \mathcal{H}_{div}$ is the *Oseen* operator
- $\Pi B \colon \mathbb{R}^2 \to \mathcal{H}_{div}$ is the input operator
- $C \colon \mathcal{H}_{div} \to \mathbb{R}^q$ is the output operator

[1] The pressure p_δ is gone, since Π maps along the orthogonal complement of the gradient

Boundedness of the input operator

Lemma (JH'21, Benner&JH'18)

If $g_i \in H_{00}^{1/2}(\Gamma_i)^2$, $i = 1, 2$, and $\varepsilon > 0$, then the input operator $B \colon \mathbb{R}^2 \to L^2(\Omega)$ for the Oseen system that realizes

$$V = n g_i u_i - \varepsilon(\nu \frac{\partial V}{\partial n} - nP) \quad \text{on } \Gamma_i, \quad i = 1, 2$$

is bounded.

Outline of the proof:

- By definition $B = \Pi B$, with Π being the orthogonal projector onto $\mathcal{H}_{div}$.
- We show that $\langle \Pi B u, w \rangle_{L^2(\Omega)} = \langle B u, \Pi w \rangle_{L^2(\Omega)}$.
- Thus, $\langle B u, w \rangle_{L^2(\Omega)} = -\frac{1}{\varepsilon} \sum_{i=1,2} \int_{\Gamma_i} \Pi w \cdot (g_i n) \, \mathrm{ds} \, u_i$.
- Since $\Pi w \cdot n \in H^{-1/2}(\Gamma_i)$, it follows $B = \Pi B \colon \mathbb{R}^2 \to L^2(\Omega)$ that.

[2] $H_{00}^{1/2}(\Gamma_i)$ contains those functions out of $H^{1/2}(\Gamma_i)$ that are boundedly extendable by 0 to the complete boundary.

Oseen Equations as Linear System

✓ The linearized model is a standard (A, B, C) system

- we know: A_α is the generator of a C_0-semi group [Raymond'06]
- we choose: C to be bounded
- we have just shown: ΠB is bounded.

➙ The theory for robust stabilization of linearization errors applies.

← Assume that the linearization point v_α is uncertain

- that is $v_\alpha \leftarrow v_\alpha + \delta_v$
- then A is perturbed $A \leftarrow A + \delta_A$
- as is the transferfunction

$$G_\delta(s) = C(sI - A - \delta_A)^{-1}B$$

Oseen Equations as Linear System

Theorem (JH'19)

Consider the perturbed Oseen system and let $L \in \mathcal{L}(\mathbb{R}^k, V^0)$ and $\delta_A(\delta_v)$ be such that $(A + \delta_A - LC)$ is exponentially stable for all δ_A small. Then the associated transferfunction G_δ has a coprime factorization

$$G_\delta = [N + \delta_N][M + \delta_M]^{-1},$$

where $NM^{-1} = G$ is the transferfunction associated with the unperturbed system, and

$$\|\delta_N\|_{H_\infty} \to 0 \quad \text{and} \quad \|\delta_M\|_{H_\infty} \to 0$$

as $\delta_v \to 0$.

1. The perturbation δ_N has the representation[3]

$$\delta_N(s) = C\delta_A(sI - A + LC)^{-1}(sI - A - \delta_A + LC)^{-1}\Pi B,$$

2. and can be realized as a cascaded system

$$\begin{aligned} \dot{v}_1 &= (A + \delta_A - LC)v_1 + \Pi B u, && (\mathcal{F}_1) \\ \dot{v}_2 &= (A - LC)v_2 + v_1 && (\mathcal{F}_2) \\ y &= C\delta_A v_2, \end{aligned}$$

   in the time domain.

3. This results in the transferfunction (in the time domain):

$$y = C\delta_A \mathcal{F}_2 \mathcal{F}_1 u.$$

Linearization error as CFP – Outline of the proof: II

For the transfer function in the time domain

$$y = C\delta_A \mathcal{F}_2 \mathcal{F}_1 u$$

we have that:

1. Certainly $\|C\delta_A\| \to 0$ if $\|\delta_A\| \to 0$, but only on function spaces with sufficient regularity. (The operator δ_A contains spatial derivatives)
2. Therefore, we use
   - the uniform stability of $A + \delta_A - LC$
   - and the analyticity of the semi-group that is generated by $A - LC$

   to show that $\mathcal{F}_2\mathcal{F}_1$ provides the needed regularity.

Linearization error as CFP – Outline of the proof: III

3. By means of a classical result[4], that connects frequency- and time domain, we infer that dass

$$\|\delta_N\|_{H_\infty} \leq \|C\delta_A \mathcal{F}_2 \mathcal{F}_1\|_{L^2 \to L^2},$$

so that $\|\delta_A\| \to 0$ implies that

$$\|\delta_N\|_{H_\infty} \to 0.$$

□

[3] Benner&JH(2016) *IFAC PapersOnLine*
[4] Weiss(1991) *Representation of shift-invariant operators on L^2 by H^∞ transfer functions*

Outline

Conclusions

Summary

Robust controller

- can compensate model uncertainties if
- they qualify as a coprime factor perturbation.

The general ∞-dimensional theory

- applies to control of incompressible flows
- if Dirichlet control is relaxed as Robin control.

Uncertainty in the linearization

- is, in fact, a coprime factor perturbation
- that smoothly depends on size of the error.

Outlook

- Quantify the error in the factorizations.
- Incorporate the discretization error in the controller design.

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