MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN SYSTEMS AND CONTROL THEORY

Numerical Methods in Control and Optimization of Dynamical Systems

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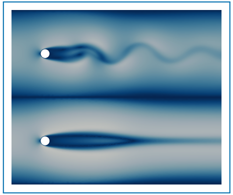
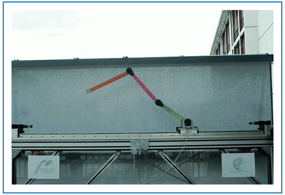
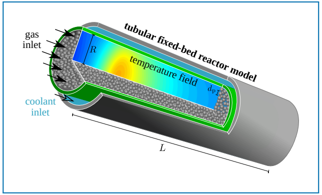
MATHEMATISCHE KOMPLEXITÄTSREDUKTION

Control





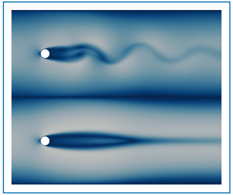


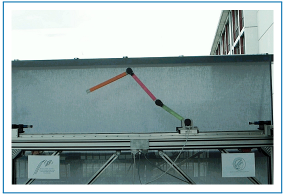
Optimization

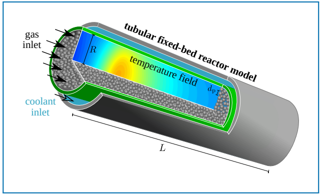
- Stabilization of
- Small Deviations
- Linear Theory
- General approaches

- Large changes
- Nonlinear methods
- Data-driven approaches

Control







Optimization

- Stabilization of
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- Linear Theory
- General approaches

Optimal Control

- Large changes
- Nonlinear methods
- Data-driven approaches

Overview

Typical Situation

- Fry a steak
- The cook controls the heat at the fireplace
- and observes the process, e.g. via measuring the temperature in the inner

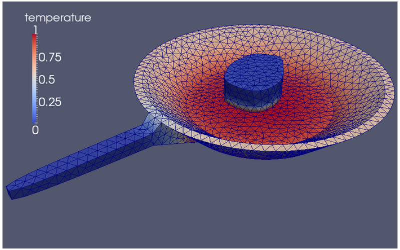
Typical Situation

- The model

$$
\begin{aligned}
\dot{\theta} &= \nabla \cdot (\nu \nabla \theta) && \text{in } (0, \infty) \times \Omega, \\
\theta &= u, && \text{at the hob}, \\
\theta(0) &= 0.
\end{aligned}
$$

- The cook controls the heat at the fireplace, which we denote by u
- and observes the process, e.g. he measures the temperature y in the center: $y = f(\theta)$.

Simulation



- The model:

$$\dot{\theta} = \nabla \cdot (\nu \nabla \theta),$$
$$\theta = u, \qquad \text{(at the hob)},$$
$$\theta(0) = 0.$$

- The cook controls the heating u
- and observes the process via $y = f(\theta)$.

- A *Finite Element* discretization [GAUL'13] of the problem leads to the finite dimensional model

$$\dot{\theta}(t) = A\theta(t) + Bu(t), \quad \theta(0) = 0,$$
$$y(t) = C\theta(t),$$

a linear time invariant (LTI) system.

Linear State Space System

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t) + Du(t),$$

with

- $x(t) \in \mathbb{R}^n$: the system's state
- $u(t) \in \mathbb{R}^m$: the input or control
- $y(t) \in \mathbb{R}^q$: the output or measurements
- $n,\ m,\ q \in \mathbb{N}$: the system dimensions

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with

- $A \in \mathbb{R}^{n \times n}$: the system matrix
- $B \in \mathbb{R}^{n \times m}$: the input matrix
- $C \in \mathbb{R}^{q \times n}$: the output matrix
- $D \in \mathbb{R}^{q \times n}$: the throughput

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Stability

Definition (Exponential Stability and Stabilizability)

- A (possibly nonlinear) dynamical system can be called exponentially stable, if all solutions x (starting in a neighborhood of the origin), decay to the origin exponentially, i.e.

$$\|x(t)\| \leq Me^{-\lambda t}\|x(0)\|, \quad t > 0,$$

for some constants M, $\lambda > 0$.

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  for some constants M, $\lambda > 0$.
- The LTI system $\dot{x} = Ax + Bu$ is called stable, if

$$\|e^{tA}\| \leq Me^{-\lambda t}, \quad t > 0.$$

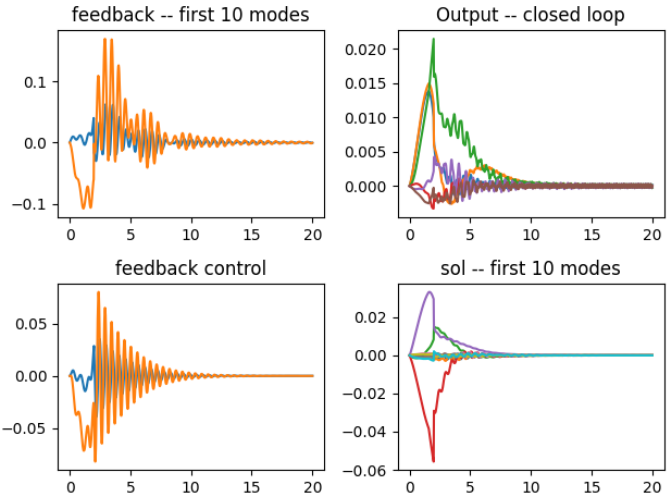
- The LTI system is called stabilizable, if there exists $K \in \mathbb{R}^{m\times n}$ such that

$$\|e^{t(A-BK)}\| \leq Me^{-\lambda t}, \quad t > 0.$$

Exponential Stability

$$\|x(t)\| \leq Me^{-\lambda t}\|x(0)\|, \quad t > 0,$$

- $M > 1$ captures transient behavior
- λ denotes the rate of decay.



Overview

Navier-Stokes Equations:

$$\dot v + (v \cdot \nabla)v - \nu\Delta v + \nabla p = \mathcal{B}u,$$
$$\nabla \cdot v = 0$$

controlled by

LTIs:

$$\begin{aligned} \dot x(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

- nonlinear
- ∞-dimensional
- accurate

$\longleftrightarrow$

- not nonlinear
- not ∞-dimensional
- not accurate

Linear vs. Nonlinear

How can linear theory work for nonlinear systems?

The self-referential promise of linear control theory

the linear controller works

$$\leftrightarrow$$

everything is under control

Set point stabilization stabilization of a nonlinear system:

$$\dot{x} = f(x) + Bu$$

Linearization about $x_* = 0$ with $f(x_*) = 0$:

$$\dot{x} = f(0) + D_x f(0)x + r(x) + Bu = A_* x + Bu + r(x)$$

with

- $A_* = D_x f(0)$ – the Jacobian of f at 0
- $r(\cdot) \in o(\|\cdot\|)$, i.e. $\lim_{x\to 0} \frac{\|r(x)\|}{\|x\|} = 0$

Theorem

If the linearized (around $x_* = 0$) problem can be stabilized, then the nonlinear problem can be stabilized locally around $x_* = 0$.

Proof:

- Let $A_* - BK$ be stable, with $e^{t(A_*-BK)} \leq Me^{-\lambda t}$, and set $u = -Kx$.
- By the *Variation of Constant* formula we have

$$x(t) = e^{t(A_*-BK)}x(0) + \int_0^t e^{(t-s)(A_*-BK)}r(x(s))\,\mathrm{ds}.$$

- By $r \in o$, for any $\eta > 0$, there exists $\delta(\eta) > 0$ such that $\|r(x)\| < \eta\|x\|$, for $\|x\| < \delta(\eta)$
- and with *Gronwall inequality*, we arrive at the inequality

$$\|x(t)\| \leq \|x(0)\| Me^{-(\lambda-\eta M)t}$$

that holds for $t > 0$ ... ... as long as $\|x(t)\| < \delta(\eta)$.

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Now

1. by continuity of x, for $x(0) < \delta(\eta)$, we have $x(0+\varepsilon) < \delta(\eta)$ – this inequality is not void!
2. choose $\eta < \frac{\lambda}{M}$, then $\lambda - \eta M > 0$ and $e^{-(\lambda-\eta M)t} < 1$ and exponentially decaying
3. choose $x(0)$, with $\|x(0)\| < \frac{\delta(\eta)}{M}$ (note that $M > 1$), so that

$$\|x(t)\| \leq \|M\| \|x(0)\| e^{-(\lambda-\eta M)t} < \delta(\eta) e^{-(\lambda-\eta M)t}$$

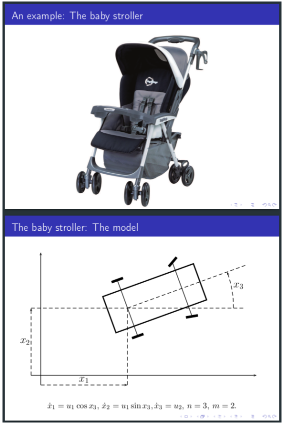
for $t > 0$. $\square$

Some side remarks

- Linearized stability sufficient for nonlinear (local) stabilizability
- Similar things hold for ∞-dimensional systems (aka control of nonlinear PDEs); [Raymond'06, Breiten&Kunisch'14].
- What if there is no linearization?
  - In finite dimensions – f not even Lipshitz – locally nonunique solutions.
  - Existence of a suitable linearization is a general problem in ∞-dimensions.

Some side remarks

- Linearized stability sufficient for nonlinear (local) stabilizability
- Similar things hold for ∞-dimensional systems (aka control of nonlinear PDEs); [Raymond'06, Breiten&Kunisch'14].
- What if there is no linearization?
  - In finite dimensions – f not even Lipshitz – locally nonunique solutions.
  - Existence of a suitable linearization is a general problem in ∞-dimensions.
- Sometimes, better stay nonlinear: A baby stroller is (nonlinearly) controllable but not linearly stabilizable.



Lecture by [J.M. Coron'11]

Challenges

So, with a linearization A_* and with a controller K, so that the LTI

$$\dot{x} = (A_* - BK)x$$

is stable, the nonlinear system

$$\dot{x} = f(x) - BKx$$

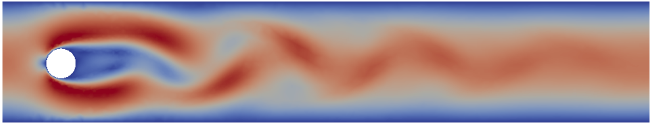
is locally stabilized around x_*.

Q: What if A_*, i.e. the linearization, is faulty?

Q: How to compute K?

Overview

Flow Control Problem

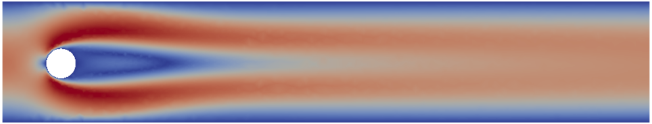


Feedback Control

Problem: The steady state is unstable: any perturbation – no matter how small – will trigger a transition into a periodic regime.

Goal: Stabilizing feedback controller that can handle:

- limited measurements,
- system uncertainties.



Flow Control Problem

Idea: Linearization-based feedback control for stabilization of the steady state.
[Raymond06, Benner&JH'15, Breiten&Kunisch'14]

$$\begin{aligned} \dot v + (v\cdot\nabla)v - \nu\Delta v + \nabla p &= Bu, \\ \nabla\cdot v &= 0 \end{aligned}$$

Linearization & Semi-Discretization

$$\begin{aligned} \dot v - Av - J^\top p &= Bu, \\ Jv &= 0 \end{aligned}$$

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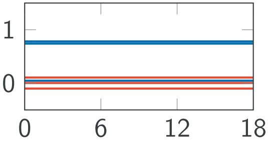
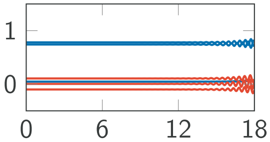
Fragility of Observer-Based Controllers

LQG controllers have no guaranteed robustness margins and will likely fail in the presence of system uncertainties.

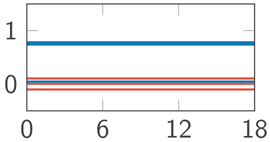
LQG-feedback



corrupted LQG-feedback



corrupted state-feedback



Introduction to Robust Control

In fact: [IEEE TRANSACTIONS ON AUTOMATIC CONTROL ('78)]:

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

***Abstract*—There are none.**

Good news: $\mathcal{H}_\infty$ controllers work under uncertainties like

- [CURTAIN'03]: Galerkin approximations of evolution systems,
- [BENNER&JH'17]: stable mixed-FEM approximation of the flow equations,
- [BENNER&JH'16]: *errors in the linearization point*,
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that can be qualified as a coprime factor perturbations.

Moreover,

- [THIS TALK, JH'21]: the coprime factor perturbation depends smoothly on the linearization error.
- [THIS TALK, BENNER&JH&WERNER'21]: we can compute the controller and its robustness

Outline

Transfer functions

Mapping of inputs (controls) to outputs (measurements) in frequency domain, i.e., after Laplace transform of the system.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad \xrightarrow{\mathcal{L}(s)} \quad \begin{aligned} sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) \end{aligned}$$

Uncertain Linearization Points are Coprime Factor Uncertainties

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

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1. A *nominal* system has the transfer function

$$G(s) = C(sI - A)^{-1}B \in \mathbb{C}^{q,r}.$$

2. But uncertainty in the operator gives another transfer function

$$G_\Delta(s) = C(sI - A - \delta_A)^{-1}B \in \mathbb{C}^{q,r}.$$

Coprime Factorization

Given a transfer function $G(s)$ of a linear system,

$$G(s) = M^{-1}(s)N(s)$$

is a **(left) coprime factorization** if there exist $X(s)$, $Y(s)$ such that the Bezout identity

$$M(s)X(s) + N(s)Y(s) = I$$

holds. Here, N, M, X, Y are all rational matrix functions with all poles in the open left half of the complex plane, i.e., they all represent stable linear systems.

Uncertain Linearization Points are Coprime Factor Uncertainties

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Coprime Factor Perturbation

$$G_\Delta(s) = [N(s) + \delta_N(s)]\,[M(s) + \delta_M(s)]^{-1}(s) \approx G(s) = N(s)M^{-1}(s),$$

where $N + \delta_N, M + \delta_N$ are stable.

Uncertain Linearization Points are Coprime Factor Uncertainties

Next we will show that

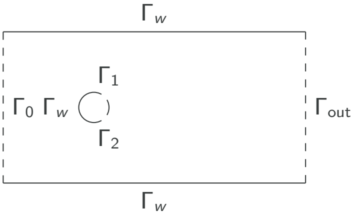
- Inexact linearizations of incompressible Navier-Stokes equations
- can be qualified as a coprime factor uncertainty
- that smoothly depends on the linearization error.

So that the standard $\mathcal{H}_\infty$-theory for robust controller design applies.

Outline

Linearized Navier-Stokes Equations as Linear System

We consider



$$
\begin{aligned}
\dot V + (V \cdot \nabla)V + \nabla P - \nu \Delta V &= 0,\\
\operatorname{div} V &= 0, \quad \text{in } \Omega,
\end{aligned}
$$

$$
\begin{aligned}
\nu \frac{\partial V}{\partial n} - nP &= 0 \text{ on } \Gamma_{out},\\
V &= 0 \text{ on } \Gamma_w,\\
V &= ng_0 \cdot \alpha \text{ on } \Gamma_0,\\
V &= ng_1 \cdot u_1 \text{ on } \Gamma_1,\\
V &= ng_2 \cdot u_2 \text{ on } \Gamma_2,
\end{aligned}
$$

where

- V ...velocity,
- P ...pressure,
- ν ...diffusion parameter,
- g_0, g_1, g_2 ...spatial shape functions,
- u_1, u_2 ...scalar input functions,
- α ...magnitude of the inflow velocity,
- n ...normal vector at the boundaries.

Linearized Navier-Stokes Equations as Linear System

A linearized I/O model is obtained as follows:

1. We relax the Dirichlet control $V\big|_{\Gamma_1} = ng_1 u - \varepsilon(\nu \frac{\partial V}{\partial n} - Pn)$
2. Let v_α be the steady state solution for zero inputs,
   and let $v_\delta(t) = V(t) - v_\alpha$ the deviation.
3. We consider the linearization

$$\dot{v}_\delta + (v_\delta \cdot \nabla)v_\alpha + (v_\alpha \cdot \nabla)v_\delta + \nabla p_\delta - \nu \Delta v_\delta = 0$$

   that is a valid approximation as long as v_δ is small.

Then, with

$$\mathcal{H}_{div} := \{v \in L^2(\Omega) : \operatorname{div} v = 0, v \cdot n = 0 \text{ on } \Gamma_w \cap \Gamma_{\text{out}}\}$$

as the state space, the (orthogonal) *Leray*-projector

$$\Pi \in \mathcal{L}(L^2(\Omega))\colon L^2(\Omega) \mapsto \mathcal{H}_{div},$$

and $x := \Pi v_\delta$ the model reads[1]

$$\begin{aligned} \dot{x} &= A_\alpha x + \Pi B u \quad \text{ in } \mathcal{H}_{div}, \\ y &= Cx \end{aligned}$$

where

- $A_\alpha\colon \mathcal{D}(A_\alpha) \subset \mathcal{H}_{div} \to \mathcal{H}_{div}$ is the *Oseen* operator
- $\Pi B\colon \mathbb{R}^2 \to \mathcal{H}_{div}$ is the input operator
- $C\colon \mathcal{H}_{div} \to \mathbb{R}^q$ is the output operator

[1] The pressure p_δ is gone, since Π maps along the orthogonal complement of the gradient

Boundedness of the input operator

Lemma (JH'21, Benner&JH'18)

If $g_i \in H^{1/2}_{00}(\Gamma_i)^2$, $i = 1, 2$, and $\varepsilon > 0$, then the input operator $B \colon \mathbb{R}^2 \to L^2(\Omega)$ for the Oseen system that realizes

$$V = n g_i u_i - \varepsilon\left(\nu \frac{\partial V}{\partial n} - nP\right) \quad \text{on } \Gamma_i, \quad i = 1, 2$$

is bounded.

Outline of the proof:

- By definition $B = \Pi B$, with Π being the orthogonal projector onto $\mathcal{H}_{div}$.
- We show that $\langle \Pi B u, w \rangle_{L^2(\Omega)} = \langle B u, \Pi w \rangle_{L^2(\Omega)}$.
- Thus, $\langle B u, w \rangle_{L^2(\Omega)} = -\frac{1}{\varepsilon} \sum_{i=1,2} \int_{\Gamma_i} \Pi w \cdot (g_i n) \, \mathrm{ds} \, u_i$.
- Since $\Pi w \cdot n \in H^{-1/2}(\Gamma_i)$, it follows that $\Pi B \colon \mathbb{R}^2 \to L^2(\Omega)$ is bounded.
- By definition $\Pi B = B$.

[2] $H^{1/2}_{00}(\Gamma_i)$ contains those functions out of $H^{1/2}(\Gamma_i)$ that are boundedly extendable by 0 to the complete boundary.

Linearized Navier-Stokes Equations as Linear System

✓ The linearized model is a standard (A, B, C) system

- we know: A_α is the generator of a C_0-semi group [Raymond'06]
- we choose: C to be bounded
- we have just shown: ΠB is bounded.

➙ The theory for robust stabilization of linearization errors applies.

← Assume that the linearization point v_α is uncertain

- that is $v_\alpha \leftarrow v_\alpha + \delta_v$
- then A is perturbed $A \leftarrow A + \delta_A$
- as is the transferfunction

$$G_\delta(s) = C(sI - A - \delta_A)^{-1}B$$

Linearized Navier-Stokes Equations as Linear System

Theorem (JH'21)

Consider the perturbed Oseen system and let $L \in \mathcal{L}(\mathbb{R}^k, V^0)$ *and* $\delta_A(\delta_v)$ *be such that* $(A + \delta_A - LC)$ *is exponentially stable for all* δ_A *small. Then the associated transferfunction* G_δ *has a coprime factorization*

$$G_\delta = [N + \delta_N][M + \delta_M]^{-1},$$

where $NM^{-1} = G$ *is the transferfunction associated with the unperturbed system, and*

$$\|\delta_N\|_{\mathcal{H}_\infty} \to 0 \quad \text{and} \quad \|\delta_M\|_{\mathcal{H}_\infty} \to 0$$

as $\delta_v \to 0$.

Linearized Navier-Stokes Equations as Linear System

Theorem (JH'21)

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as $\delta_v \to 0$.

Remark on the existence of L

The existence of the uniformly stabilizing L (as we have assumed it here) is much less critical then the existence of a robust controller (because the L is a *state feedback*).

1. The perturbation δ_N has the representation[3]

$$\delta_N(s) = C\delta_A(sI - A + LC)^{-1}(sI - A - \delta_A + LC)^{-1}\Pi B,$$

2. and can be realized as a cascaded system

$$\begin{aligned} \dot v_1 &= (A + \delta_A - LC)v_1 + \Pi B u, && (\mathcal{F}_1)\\ \dot v_2 &= (A - LC)v_2 + v_1 && (\mathcal{F}_2)\\ y &= C\delta_A v_2, \end{aligned}$$

   in the time domain.

3. This results in the transferfunction (in the time domain):

$$y = C\delta_A \mathcal{F}_2 \mathcal{F}_1 u.$$

Linearization error as CFP – Outline of the proof: II

For the transfer function in the time domain

$$y = C\delta_A\mathcal{F}_2\mathcal{F}_1 u$$

we have that:

1. Certainly $\|C\delta_A\| \to 0$ if $\|\delta_A\| \to 0$, but only on function spaces with sufficient regularity. (The operator δ_A contains spatial derivatives)
2. Therefore, we use
   - the uniform stability of $A + \delta_A - LC$
   - and the analyticity of the semi-group that is generated by $A - LC$

   to show that $\mathcal{F}_2\mathcal{F}_1$ provides the needed regularity.

3. By means of a classical result[4], that connects frequency- and time domain, we infer that dass

$$\|\delta_N\|_{\mathcal{H}_\infty} \leq \|C\delta_A \mathcal{F}_2 \mathcal{F}_1\|_{L^2 \to L^2},$$

so that $\|\delta_A\| \to 0$ implies that

$$\|\delta_N\|_{\mathcal{H}_\infty} \to 0.$$

□

[3] Benner&JH(2016) *IFAC PapersOnLine* based on the textbook by Curtain&Zwart(1995)
[4] Weiss(1991) *Representation of shift-invariant operators on L^2 by H^∞ transfer functions*

How can this help?

One can show [Mustafa&Glover'91, Benner/JH/Werner'19, Zhou/Doyle/Glover'96]:

- that uncertainty in the linearization can be formulated as an *normalized $\mathcal{H}_\infty$ robust control problem*
- and that $\mathcal{H}_\infty$ robust controller K of robustness margin γ will stabilize the perturbed system if

$$\left\| \begin{bmatrix} \delta_N & \delta_M \end{bmatrix} \right\|_{\mathcal{H}_\infty} < \gamma^{-1}.$$

So let's compute such a controller...

Outline

Flow Control Problem

$\mathcal{H}_\infty$ Riccati Equations [Zhou/Doyle/Glover'95]

Given some simplifying assumptions, there exists an admissible controller $K(s) \iff$:

1. There exists a stabilizing solution $X_\infty = X_\infty^\top \geq 0$ to the regulator Riccati equation
$$A^\top X_\infty + X_\infty A + C^\top C - (1-\gamma^{-2})X_\infty BB^\top X_\infty = 0.$$
2. There exists a stabilizing solution $Y_\infty = Y_\infty^\top \geq 0$ to the filter Riccati equation
$$AY_\infty + Y_\infty A^\top + BB^\top - (1-\gamma^{-2})Y_\infty C^\top CY_\infty = 0.$$
3. It holds $\gamma^2 > \lambda_{\max}(Y_\infty X_\infty)$.

$\mathcal{H}_\infty$ Riccati Equations [Zhou/Doyle/Glover'95]

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3. It holds $\gamma^2 > \lambda_{\max}(Y_\infty X_\infty)$.

The central (or minimum entropy) controller $\hat{K}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ is given by

$$\hat{A} = A - (1-\gamma^{-2}) B B^\top X_\infty - Z_\infty Y_\infty C^\top C, \quad \hat{B} = Z_\infty Y_\infty C^\top, \quad \hat{C} = -B^\top X_\infty,$$

with $Z_\infty = (I_n - \gamma^{-2} X_\infty Y_\infty)^{-1}$.

Arising Challenges

Large-Scale Matrix Equations

How to solve the arising large-scale sparse Riccati equations

$$A^\top X_\infty + X_\infty A + C_1^\top C_1 + X_\infty(\gamma^{-2} B_1 B_1^\top - B_2 B_2^\top) X_\infty = 0?$$

- **Low-rank Riccati iteration** solves indefinite Riccati equations by an approximation

$$X_\infty \approx ZZ^\top,$$

with $Z \in \mathbb{R}^{n \times r}$ and $r \ll n$. [Lanzon/Feng/Anderson '07, Benner/Heiland/W. '22a]

$\mathcal{H}_\infty$ Robust Controller and Riccati Equations

Arising Challenges

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High Dimensional Controller

How to construct a low dimensional $\hat{K}(s)$ for faster evaluation?

- Use model order reduction based on X_∞ and Y_∞.

Notation:

- normalized left coprime factorizations $G = M^{-1}N$, $G_r = M_r^{-1}N_r$ (for computation see [Benner/Heiland/W. '19]),
- $\beta = \sqrt{1-\gamma^{-2}}$.

The approximation error of the $\mathcal{H}_\infty$ balanced truncation is given by

$$\left\| \begin{bmatrix} \beta(N - N_r) & M - M_r \end{bmatrix} \right\|_{\mathcal{H}_\infty} =: \beta\hat{\epsilon} \leq \beta\epsilon = 2 \sum_{k=r+1}^{n} \frac{\beta\sigma_k^{\mathcal{H}_\infty}}{\sqrt{1+\beta^2\left(\sigma_k^{\mathcal{H}_\infty}\right)^2}},$$

where $\sigma_k^{\mathcal{H}_\infty}$ are the $\mathcal{H}_\infty$ *characteristic values*.

(Still) Stabilizing Reduced-Order Controller

Notation:

- normalized left coprime factorizations $G = M^{-1}N$, $G_r = M_r^{-1}N_r$ (for computation see [Benner/Heiland/W. '19]),
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where $\sigma_k^{\mathcal{H}_\infty}$ are the $\mathcal{H}_\infty$ *characteristic values*.

Theorem [Mustafa/Glover '91]

The reduced-order $\mathcal{H}_\infty$ controller is guaranteed to stabilize the full-order system if

$$\hat{\epsilon}(\beta+\gamma) < 1 \quad \text{or} \quad \epsilon(\beta+\gamma) < 1.$$

Numerical Realization of the DAE Structure

For consistent initial values, i.e., $Jv_0 = 0$, the semi-discretized Navier-Stokes equation can be realized by an ODE system:

$$\begin{aligned} E\dot{v} &= Av + J^\top p + Bu, \\ 0 &= Jv, \\ y &= Cv, \end{aligned} \quad \Longrightarrow \quad \begin{aligned} E\dot{v} &= \Pi^\top A \Pi v + \Pi^\top B, \\ y &= C\Pi v, \end{aligned}$$

where $\Pi = I_{n_v} - E^{-1}J^\top (JE^{-1}J^\top)^{-1}J$ is the discrete Leray projection.

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Implicit Realization [Heinkenschloss&Sorensen&Sun '08]

The explicit projection Π can be avoided in the numerical methods by solving saddle point problems of the type

$$\begin{bmatrix} A + s_i E & J^\top \\ J & 0 \end{bmatrix} \begin{bmatrix} X \\ * \end{bmatrix} = \begin{bmatrix} Y \\ 0 \end{bmatrix}.$$

Linearization Uncertainties

In general, an uncertainty $\mathcal{A}_\Delta$ in the linearization $\mathcal{A}$ ...

$$\begin{aligned}\mathcal{E}\dot{x}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t),\\ y(t) &= \mathcal{C}x(t)\end{aligned} \quad \Rightarrow \quad \begin{aligned}\mathcal{E}\dot{x}(t) &= [\mathcal{A} + \mathcal{A}_\Delta]x(t) + \mathcal{B}u(t),\\ y(t) &= \mathcal{C}x(t)\end{aligned}$$

... is an additive uncertainty in the transfer function

$$G(s) = \mathcal{C}(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{B} \quad \Rightarrow \quad \begin{aligned}G_\Delta(s) &= \mathcal{C}(s\mathcal{E} - \mathcal{A} - \mathcal{A}_\Delta)^{-1}\mathcal{B}\\ &= G(s) + \tilde{G}(s)\end{aligned}$$

where $\tilde{G}(s) = \mathcal{C}\mathcal{A}_\Delta(s\mathcal{E} - \mathcal{A})^{-1}(s\mathcal{E} - \mathcal{A} - \mathcal{A}_\Delta)^{-1}\mathcal{B}$.

Linearization Uncertainties

In general, an uncertainty $\boldsymbol{\mathcal{A}_\Delta}$ in the linearization $\mathcal{A}$ ...

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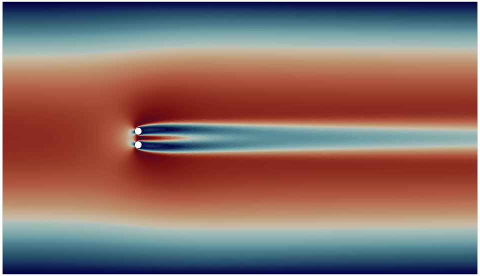
where $\tilde{G}(s) = \mathcal{C}\boldsymbol{\mathcal{A}_\Delta}(s\mathcal{E} - \mathcal{A})^{-1}(s\mathcal{E} - \mathcal{A} - \boldsymbol{\mathcal{A}_\Delta})^{-1}\mathcal{B}$.

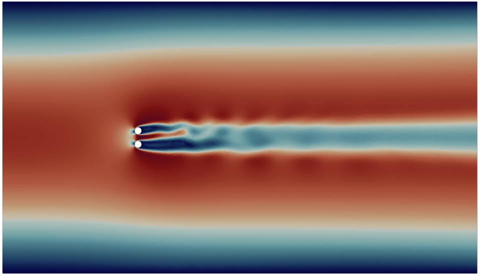
Additive uncertainties can be compensated by robust $\mathcal{H}_\infty$ controller design.

Double Cylinder

Setup: Stabilization of the steady state [Benner&JH&Werner'21]



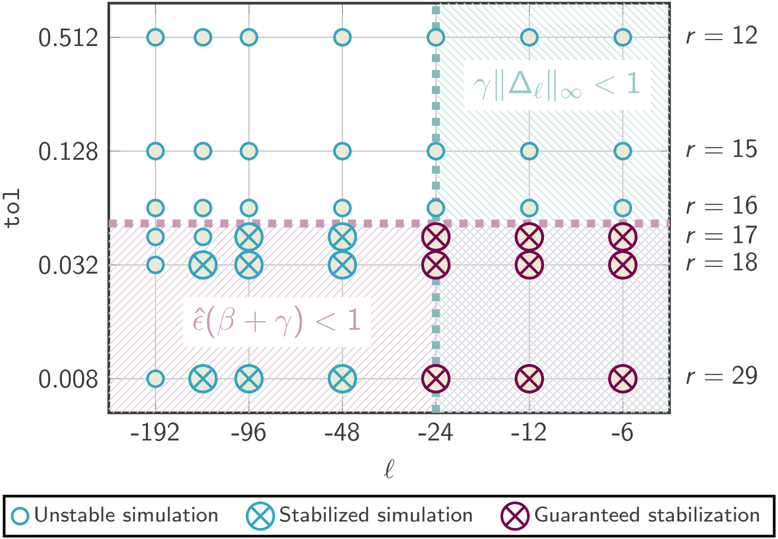
(a) Steady state.



(b) Natural flow.

- Navier-Stokes equations discretized by Taylor-Hood finite elements
- system order $n = 51\,337$
- Reynolds number 60
- boundary control: individual rotation of both cylinders
- observations: 3 velocity sensors in the wake behind the cylinders

Double-cylinder: Results [Benner/Heiland/W. '22]



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