Nonlinear Feedback Stabilization of Incompressible Flows via Updated Riccati-based Gains

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Abstract— **The stabilization or set point control of incompressible laminar flows has been under vivid investigation since long. All linearization based approaches suffer from the conceptual shortcoming of a possibly small domain of attraction. In order to get the system into the regime where, e.g., Riccati-based feedback stabilization works, nonlinear control laws are necessary. Therefore, we propose a scheme that continuously updates an initial feedback, that guarantees decay of solutions under locally checkable conditions, and that can be realized through solving large-scale linear equations.**

I. Introduction

In the last decade, stabilization of incompressible flows on the base of stationary Riccati-based feedback has been investigated in theory [17], [18] and in numerical simulations [1], [2], [7], [12]. In view of applications, the Riccati-based approach, however, has the conceptual shortcoming that it assumes the flow in the vicinity of the target state at starting time. Since the target is an unstable steady state, this initial condition is unlikely to be fulfilled in practice. Thus, it is necessary to first transfer the system from a possible into a desired state before Riccati-based feedback stabilization can be applied; cf. [4].

To force the system into the desired state, a corresponding control that solves the set point control problem has to be found and applied. With the practical realization in mind, we limit our considerations to closed-loop approaches that will provide the control in feedback form, leaving aside all work on open-loop control of flows. In view of generality, we focus on universal approaches and only mention the alternatives that base on heuristic low-order models as in [15] here. Similarly, we will leave aside the recent result on global feedback stabilization of Navier–Stokes equations [16] since it bases on eigenfunctions which may be infeasible to compute.

The most common nonlinear feedback set point control methods for flow equations base on solving local optimization problems. The *instantaneous control* algorithm [9], [13] computes a control from a local prediction of the system evolution. *Receding horizon* or *Model Predictive Control* (MPC) approaches as used in [6] use larger prediction horizons. Other approaches resort to approximate solutions of *HJB* equations; cf. [14].

Instantaneous control for the Navier–Stokes equations (NSE) has been shown capable [13] to drive a system to a set point, if the initial deviation is small. The MPC formulations have not been analysed for the NSE in particular. General results [11] on the stabilization through MPC, assume sufficiently large control horizons or suitably chosen *terminal conditions* for the local optimization.

In this paper, we propose the adaptation of a recently developed [5] update scheme to semi-discrete NSE. This scheme provides continuous updates of an initial Riccati-based feedback-gain which can guarantee exponential decay of the trajectory of an autonomous nonlinear system.

The paper is organized as follows. In Section II, we shortly revisit the proposed scheme. In Section III, we extend the theory to the incompressible NSE and, in view of numerical implementations, reformulate the Sylvester and Riccati equations that define the gain updates as a matrix valued saddle-point problem. In Section IV, we expand on how the matrix equations could be solved. We conclude with a general discussion on the expectations and limitations of the approach.

II. Nonlinear Feedback via Continuous Updates

We consider an autonomous nonlinear system with affine inputs

$$\dot{\zeta}(t) = f(\zeta(t)) + Bu(t), \quad \zeta(0) = z_0 \in \mathbb{R}^n, \tag{1}$$

with a function f and a matrix B of appropriate dimensions. The task is to find an input u, with $u(t) \in \mathbb{R}^k$, that forces the system (1) into a set point $z^* \in \mathbb{R}^n$, i.e. a system state z^* for which $f(z^*) = 0$. Under mild assumptions, cf. [5], this task can be accomplished by finding u as an input that drives the solution trajectory of the system

$$\dot{\xi}(t) = A(\xi(t))\xi(t) + Bu(t), \quad \xi(0) = x_0 \tag{2}$$

to zero, where the matrix-valued function $A\colon \mathbb{R}^n \to \mathbb{R}^{n,n}$ is defined in terms of f and where $x_0 := z_0 - z^*$. System (2) is called a *state dependent coefficient* (SDC) system or an *extended linearization*.

Let $X_0 \subset \mathbb{R}^n$ be a bounded set that contains the origin and for $T > 0$, let

$$\Xi_{[0,T]} := \big\{\xi : \xi \text{ solves (2) with } u = 0 \text{ and } x_0 \in X_0\big\} \tag{3}$$

the set that contains all solution trajectories starting in X_0 and

$$X_{[0,T]} := \big\{\xi(t) : \xi \in \Xi_{[0,T]} \text{ and } 0 \le t \le T\big\} \tag{4}$$

the set of all assumed values.

Theorem 1 ([5], Thm. 3.3 and Lem. 3.4): Consider (2) with $u = 0$ and a smoothly differentiable norm. Let

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$X_0 \in \mathbb{R}^n$ and let $T > 0$ be such that $X_{[0,T]}$ is bounded. Let A be smoothly differentiable on $X_{[0,T]}$ and let there exist constants K, $\omega > 0$ such that

$$\|e^{sA(x)}\| \leq K e^{-\omega s} \quad \text{for all } x \in X_{[0,T]} \text{ and } s > 0.$$

Let

$$m_t := \inf_{\rho \in \mathbb{R}_{\geq 0}} \sup_{\xi \in \Xi_{[0,T]}} \frac{\int_0^t e^{-\omega(t-s)} \|A(\xi(s)) - A(\xi(\rho))\| \|\xi(s)\| \, \mathrm{d}s}{\int_0^t e^{-\omega(t-s)} |s-\rho| \|\xi(s)\| \, \mathrm{d}s}.$$

If there exists a t^*, $0 \leq t^* \leq T$, such that

$$-\omega^* := \frac{\ln K}{t^*} + \sqrt{K m_{t^*} \ln 2} - \omega \tag{5}$$

is negative, then all solutions to (2), that start in X_0, decay to zero exponentially.

From the theorem and in particular from (5) it follows, that the decay of solutions can be guaranteed, if the coefficient matrix $A(\xi(s))$ is uniformly stable over a sufficiently long time interval with a decay rate ω strong enough to damp out transient effects (that basically define the constant K). Accordingly, stabilization of (2) can be achieved by a feedback law $u = -F(x)x$ that would make the SDC matrix $\tilde{A}(x) = A(x) - BF(x)$ of the closed-loop system

$$\dot{\xi}(t) = \big(A(\xi(t)) - BF(\xi(t))\big)\xi(t) \tag{6}$$

sufficiently uniformly stable in the sense of Theorem 1.

The following theorem defines a continuous update that keeps the stability constants of an initial Riccati-based feedback in a predefined margin.

Theorem 2 ([5], Thm. 4.2, Lem. 4.4): For given $x_0 \in \mathbb{R}^n$ let $\big(A(x_0), B\big)$ from (2) be stabilizable in the sense of a linear system. For given $R \in \mathbb{R}^{k,k}$ symmetric positive definite and $Q \in \mathbb{R}^{n,n}$ symmetric positive semi-definite, let $F_0 = R^{-1}B^{\mathsf{T}}P(x_0)$ where $P(x_0)$ is the unique stabilizing solution of

$$PA(x_0) + A(x_0)^{\mathsf{T}}P - PBR^{-1}B^{\mathsf{T}}P + Q = 0 \tag{7}$$

and let K, ω be the stability constants of the closed loop matrix, i.e.

$$\|e^{s(A(x_0)-BF_0)}\| \leq K e^{-\omega s}, \quad s > 0.$$

If for an $A_\Delta \in \mathbb{R}^{n,n}$, there exists $R_\Delta \in \mathbb{R}^{k,k}$ such that

$$\big(A(x_0) + A_\Delta\big)E - E\big(A(x_0) - BF_0\big) = -A_\Delta + BR_\Delta B^{\mathsf{T}}P(x_0) \tag{8}$$

has a solution E with $\|E\| < 1$, then the updated feedback gain

$$F_\Delta := (R^{-1} + R_\Delta)B^{\mathsf{T}}P(I+E)^{-1} \tag{9}$$

stabilizes $A(x_0) + A_\Delta$ and

$$\|e^{s(A(x_0)+A_\Delta - BF_\Delta)}\| \leq \tilde{K} e^{-\omega s}, \quad s > 0,$$

with $\tilde{K} = \frac{1+\|E\|}{1-\|E\|}K$.

Proof: If P is a solution to the Riccati equation (7) then it fulfills the relation

$$\begin{bmatrix} A(x_0) & -BR^{-1}B^{\mathsf{T}} \\ -Q & -A(x_0)^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} (A - BF_0).$$

For a change in the system $A(x_0) \leftarrow A(x_0) + A_\Delta$ and chosen updates in the weighing matrices $R^{-1} \leftarrow R^{-1} + R_\Delta$ and $Q \leftarrow Q + Q_\Delta$, we consider the perturbed relation

$$\begin{bmatrix} A(x_0) + A_\Delta & -B(R^{-1} + R_\Delta)B^{\mathsf{T}} \\ -Q + Q_\Delta & -A(x_0)^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} I + E \\ P \end{bmatrix} = \begin{bmatrix} I + E \\ P \end{bmatrix} (A - BF_0). \tag{10}$$

If there exists a matrix E with $\|E\| < 1$ such that (10) holds, then a multiplication of the first block equation of (10) by $(I+E)^{-1}$ from the right gives

$$A(x_0) + A_\Delta - B(R^{-1} + R_\Delta)B^{\mathsf{T}}P(I+E)^{-1} = (I+E)(A - BF_0)(I+E)^{-1},$$

or

$$A(x_0) + A_\Delta - BF_\Delta = (I+E)(A - BF_0)(I+E)^{-1},$$

which leads us to the conclusion that the updated closed-loop matrix has the same eigenvalues as the initial closed-loop matrix and, thus, is stable with the same decay rate ω. The estimate on the bound $\tilde{K}$ on the transient behavior is obtained by considering $(I+E)^{-1}$ as a *Neumann series*, which by $\|E\| < 1$ converges with $\|(I+E)^{-1}\| < \frac{1}{1-\|E\|}$. Furthermore, we note that if such an E exists, then the second block row equation of (10) uniquely defines Q_Δ such that, in fact, one only has to consider the first block row of (10), i.e., after a rearrangement of the terms, equation (8).

□

Remark 3: As defined in Theorem 2, the constant $\tilde{K}$ is unbounded. If one enforces $\|E\| \leq c < 1$ for a threshold value c, then $\tilde{K} \leq \frac{1+c}{1-c}$ is bounded.

For nonlinear feedback design for (2), Theorem 2 can be applied with $A_\Delta = A(\xi(t)) - A(x_0)$ at every time instance t, to define a gain via (9) that keeps the closed loop uniformly stable for some time. The applicability and efficiency of this design has been demonstrated for a PDE, namely the *Chafee Infantee* equation and an ODE that models a chemical reactor in [5].

III. Gain Updates for Navier–Stokes Equations

In this section, we extend the theory towards set point control of spatially discretized NSE

$$M\dot{v}(t) = N(v(t))v(t) + Lv(t) + G^{\mathsf{T}}p(t) + Bu(t), \quad v(0) = v_0, \tag{11a}$$

$$0 = Gv(t), \tag{11b}$$

with $M \in \mathbb{R}^{n,n}$, $L \in \mathbb{R}^{n,n}$, $G \in \mathbb{R}^{m,n}$, $n > m$, and $N\colon \mathbb{R}^n \to \mathbb{R}^{n,n}$ linear, that models the evolution of the velocity v and pressure p in an incompressible flow. We focus on the differential-algebraic structure of (11). For considerations on the approximation of the dynamics of flows with *Finite Elements*, we refer to [10]. Concerning the numerical realization of boundary control in (11), we refer to [3].

Let v^* be a set point of (11), i.e. there exists a p^* such that

$$0 = N(v^*)v^* + Lv^* + G^\mathsf{T} p^*,$$
$$0 = Gv^*.$$

We define $v_\delta := v - v^*$ and $p_\delta = p - p^*$ and consider the difference system

$$M\dot{v_\delta}(t) = A(v_\delta(t))v_\delta(t) + G^\mathsf{T} p_\delta(t) + Bu(t), \qquad v(0) = v_{\delta 0} := v_0 - v^*, \tag{12a}$$
$$0 = Gv_\delta(t), \tag{12b}$$

where $A(v_\delta) := N(v_\delta) + N(v^*) + N^*(v^*) + L \in \mathbb{R}^{n,n}$ and N^* is defined via $N^*(v^*)v_\delta = N(v_\delta)v^*$. For what follows, we make the assumption that M is symmetric strictly positive definite and that G has full rank. In this case, we can define the projector $\Pi := I - M^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}G$ with the properties that

$$G\Pi = 0, \quad \Pi^\mathsf{T} M = M\Pi, \quad \text{and} \quad \Pi M^{-1} = M^{-1}\Pi^\mathsf{T}. \tag{13}$$

Then, one can show [12] that v_δ is the velocity part of the solution to (12) if and only if v_δ solves

$$M\dot{v_\delta}(t) = \Pi^\mathsf{T} A(v_\delta(t))\Pi v_\delta(t) + \Pi^\mathsf{T} Bu(t), \quad v(0) = v_{\delta 0}, \tag{14}$$

provided that $Gv_{\delta 0} = 0$. In particular, if $Gv_{\delta 0} = 0$, then any solution to (14) fulfills $Gv_\delta \equiv 0$, i.e.

$$v_\delta(t) = \Pi v_\delta(t), \quad t \geq 0. \tag{15}$$

We now formulate the nonlinear feedback update scheme as it follows from Theorem 2 for the projected Navier–Stokes equations. In a second step, we derive a formulation of the matrix equations that avoids the projector Π and the matrix inverse and, thus, is more suitable for numerical approaches.

Having premultiplied equation (14) by M^{-1}, it follows directly from (7) and (8), that for suitable R and Q, a uniformly stabilizing feedback for (14) can be obtained through an initial feedback gain $F_0 = R^{-1}B^\mathsf{T}\Pi M^{-1}P_0$, where P_0 is a stabilizing solution to

$$PM^{-1}\Pi^\mathsf{T} A_0\Pi + \Pi^\mathsf{T} A_0^\mathsf{T}\Pi M^{-1}P - PM^{-1}\Pi^\mathsf{T} BR^{-1}B^\mathsf{T}\Pi M^{-1}P + Q = 0, \tag{16}$$

and a continuous update

$$F(t) := R^{-1}B^\mathsf{T}\Pi M^{-1}P_0(I + E(t))^{-1} \tag{17}$$

defined through a solution $E(t)$ to

$$M^{-1}\Pi^\mathsf{T}\big(A_0 + A_\Delta(t)\big)\Pi E - EM^{-1}\Pi^\mathsf{T}\big(A_0\Pi - BF_0\big) = -M^{-1}\Pi^\mathsf{T} A_\Delta(t)\Pi, \tag{18}$$

where $A_\Delta(t) := A(v_\delta(t)) - A_0)$ and $A_0 := A(v_{\delta 0})$ and with $R_\Delta = 0$. Thus, if $M^{-1}\Pi^\mathsf{T}\big(A(v_{\delta 0}) - BF_0\big)$ is stable with constants K and ω, then

$$\|e^{sM^{-1}\Pi^\mathsf{T}\left(A(v_\delta(t)) - BF(t)\right)}\Pi\| \leq \tilde{K}e^{-\omega s}, \quad s > 0$$

with $\tilde{K} = \frac{1+c}{1-c}K$ as long as (18) has a solution with $\|E(t)\| < c < 1$. Note that stability is to be considered only with respect to trajectories that evolve in the range of Π; cf. (15).

The following results characterize the feedback defining matrix equations and propose a reformulation which does not call on the projector or on the matrix inverse.

Lemma 4: Consider the definition (17) of the feedback law for the projected equation (14).

1) If $(M^{-1}\Pi^\mathsf{T} A_0, M^{-1}\Pi^\mathsf{T} B)$ is stabilizable, then the initial feedback can be defined via $F_0 = R^{-1}B^\mathsf{T} X_0 M$, where X_0 is a stabilizing solution to

$$A_0^\mathsf{T} XM + MXA_0 - MXBR^{-1}B^\mathsf{T} XM + MYG + G^\mathsf{T} Y^\mathsf{T} M = -Q, \tag{19a}$$
$$GXM = 0, \quad \text{and} \quad MXG^\mathsf{T} = 0, \tag{19b}$$

for a suitable $Y \in \mathbb{R}^{n,m}$.

2) The updated gains can be obtained as

$$F(t) := R^{-1}B^\mathsf{T} XM(I + Z(t)M)^{-1},$$

where $Z(t)$ is a solution to

$$\big(A_0 + A_\Delta\big)ZM - MZ(A_0 - BF_0) + MY_1G + G^\mathsf{T} Y_2^\mathsf{T} M = -A_\Delta, \tag{20a}$$
$$GZM = 0, \quad \text{and} \quad MZG^\mathsf{T} = 0, \tag{20b}$$

with $\|ZM\| < 1$ and for suitable Y_1, $Y_2 \in \mathbb{R}^{n,m}$, provided that such a solution triple (Z, Y_1, Y_2) exists.

Proof: ad *1.*: We first note that if P_0 defines a feedback for (14) via $u(t) = -R^{-1}B^\mathsf{T}\Pi M^{-1}P_0 v_\delta(t)$, then the projected part $\Pi^\mathsf{T} P_0\Pi$ defines the same feedback. In fact, by the properties (13) of Π, by $\Pi = \Pi^2$, and by (15) it follows that

$$R^{-1}B^\mathsf{T}\Pi M^{-1}P_0 v_\delta(t) = R^{-1}B^\mathsf{T}\Pi M^{-1}\Pi^\mathsf{T} P_0\Pi v_\delta(t).$$

Next, we show that the relevant part P_0, namely $\Pi^\mathsf{T} P_0\Pi$, can be obtained as $\Pi^\mathsf{T} P_0\Pi = MX_0M$, where X_0 is the stabilizing solution to (19). Here, we call X_0 stabilizing if it is a stabilizing solution to

$$\Pi^\mathsf{T} A_0^\mathsf{T}\Pi XM\Pi + \Pi^\mathsf{T} MX\Pi^\mathsf{T} A_0\Pi - \Pi^\mathsf{T} MBR^{-1}B^\mathsf{T} XM\Pi = -\Pi^\mathsf{T} Q\Pi \tag{21}$$

which is unique by assumption.

A direct comparison of (16), multiplied by Π^T and Π from the left and the right, respectively, and (21) gives that their unique solutions indeed relate like $\Pi^\mathsf{T} P_0\Pi = MX_0M$. In order to employ the unprojected system for the definition of $\Pi^\mathsf{T} P_0\Pi$, we need to show that this X_0 defines a solution to (19). Since $X_0 = M^{-1}\Pi^\mathsf{T} P_0\Pi M^{-1} = \Pi M^{-1}P_0M^{-1}\Pi^\mathsf{T}$, we conclude that $X_0 = \Pi X_0\Pi^\mathsf{T}$ and that X_0 readily fulfills the constraints (19b). If we plug $\Pi X_0\Pi^\mathsf{T}$ into (19a) and use that $\Pi X_0\Pi^\mathsf{T}$ solves (21), we are left with

$$(I - \Pi^\mathsf{T})A_0^\mathsf{T} X_0M + MX_0A_0(I - \Pi) + MYG + G^TY^TM = Q - \Pi^\mathsf{T} Q\Pi \tag{22}$$

and the task to find a Y that solves this remainder equation. We make the ansatz $Y = \Pi Y + (I-\Pi)Y := Y_0 + Y_G$ and project the remainder equation (22) onto several subspaces to compute Y_0 and Y_G seperately. If we apply $I - \Pi^\mathsf{T}$ from the left and Π from the right to (22), we obtain

$$(I-\Pi^\mathsf{T})A_0X_0M + G^\mathsf{T}Y_0M = -(I-\Pi^\mathsf{T})Q\Pi \qquad (23)$$

which uniquely defines Y_0M and, thus, Y_0, since G^T has full column rank and since $I-\Pi^\mathsf{T}$ maps into the range of G^T. Applying Π^T from the left and $I-\Pi$ from the right, gives the transpose of (23) which is redundant in this symmetric case.

If we apply $I-\Pi^\mathsf{T}$ from the left and $I-\Pi$ from the right and recall that $(I-\Pi) = M^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1}G$ we obtain the equation

$$MY_GG + G^TY_G^TM = (I-\Pi^\mathsf{T})Q(I-\Pi^\mathsf{T})$$

that has the general solution

$$Y_G = \frac{1}{2}M^{-1}(I-\Pi^\mathsf{T})QM^{-1}G^\mathsf{T}(GM^{-1}G^\mathsf{T})^{-1} + M^{-1}G^\mathsf{T}S, \qquad (24)$$

where S is an arbitrary skew-symmetric matrix of suitable size; see [8].

Thus, the solutions defined in (24), (23), and (21) constitute a solution to (19) such that the part X_0 realizes the initial feedback as

$$R^{-1}B^\mathsf{T}M^{-1}\Pi^\mathsf{T}P_0\Pi v_\delta(t) = R^{-1}B^\mathsf{T}X_0Mv_\delta(t).$$

ad *2.*: We show that if Z, together with suitable Y_1, Y_2, solves (20) and $\|ZM\| < 1$, then $E_Z := ZM$ solves (18). By (20b) it follows that $Z = \Pi Z \Pi^T$ and $E_Z = \Pi E_Z \Pi$. If one replaces ZM by $\Pi E_Z \Pi$ in (20a), one finds that E_Z solves

$$\big(A_0 + A_\Delta\big)\Pi E_Z\Pi - M\Pi E_Z\Pi M^{-1}(A_0 - BF_0) + MY_1G + G^\mathsf{T}Y_2^\mathsf{T}M = -A_\Delta.$$

Having multiplied this equation by $M^{-1}\Pi^T$ and Π from the left and the right, respectively, one obtains that E_Z solves

$$M^{-1}\Pi^\mathsf{T}\big(A_0 + A_\Delta\big)\Pi E_Z\Pi - \Pi E_Z\Pi M^{-1}(A_0 - BF_0)\Pi = -M^{-1}\Pi^\mathsf{T}A_\Delta\Pi,$$

which is equivalent to (18) since, in particular, with $P_0 = \Pi^\mathsf{T}P_0\Pi$, it holds that $F_0\Pi = F_0$ and, by construction, it holds that $E_Z\Pi = E_Z$. □

Lemma 4 provides constituting equations for the feedback that, because they are formulated in the original coefficients of (11), seem favorable for numerical approaches. Nonetheless, the solutions to them might not be unique or might not exist. In the following remarks, we address some issues concerning practical application.

Remark 5:

1) If $Q = \Pi^\mathsf{T}Q\Pi$, then every solution X to (19) defines a solution $P = MXM$ to (16).
2) If defined through a solution to the constrained equations (19), the irrelevant part $P_0 - \Pi^TP_0\Pi$ of P_0 is zero. Accordingly, a stabilizing solution is uniquely defined through (19), which might not be the case for solutions to (16).
3) In this sense, also the solution to (20) is less ambiguous if compared to the solution of (18). In fact, with $P_0 = \Pi^\mathsf{T}P_0\Pi$, only the part $\Pi E\Pi$ is relevant for the feedback. This can be seen with the formula $\big(I+E\big)^{-1} = \sum_{i=1}^{\infty}(-E)^i$, which is valid for E with $\|E\| < 1$, and the arguments used in part *1.* of the proof of Lemma 4:
$$P_0\big(I+E\big)^{-1}v_\delta(t) = \Pi^\mathsf{T}P_0\Pi\sum_{i=1}^{\infty}(-E)^i\Pi v_\delta(t) = \Pi^\mathsf{T}P_0\Pi\big(I + \Pi E\Pi\big)^{-1}v_\delta(t).$$
4) Nevertheless, the existence of solutions to (20) is similarly unclear as for (8); cf. the discussion in [5].

IV. On the Numerical Realization of the Feedback Updates

In order to realize the nonlinear feedback in a simulation, one first has to solve a projected or constrained Riccati equation (16) or (19) and then, in every time step, the projected or constrained Sylvester equation (18) or (20). If Q is of low-rank, as it typically is the case, one can call on efficient low-rank iteration solvers, cf., e.g., [1] and [12, Ch. 9.], for the numerical solution of the Riccati equation. The Sylvester equations for the updates, however, do not have a low-rank structure. Although it is a linear equation with sparse and, in the case of a small number of inputs, low-rank coefficients, a direct solution is not feasible, since its solution will generically be a dense N by N matrix, where N is the dimension of the state-space. And in the interesting case of unstable flows, the state-space dimension N is of order 10^5. Moreover, the right-hand side A_Δ is unstructured.

Thus, for the realization of this nonlinear feedback approach, one first needs a memory efficient approach to large-scale unstructured Sylvester equations. Possible approaches may base on sparse best-approximations as they are used in *compressed sensing*.

V. Conclusion

We have proposed a nonlinear feedback design approach for semi-discrete incompressible Navier–Stokes equations. The approach comes with sufficient conditions for stability of the nonlinear closed-loop system, that can be locally estimated and cast into an algorithm for numerical realization. Another theoretical advantage of the approach is its generality and that only requires the solution of linear systems. For this purpose, we have shown that the feedback is equivalently defined through matrix equations in the original coefficients such that, e.g., structure and sparsity are preserved. However, in its current form, the algorithm requires the solution of large-scale unstructured Sylvester equations, for which there are no efficient algorithms. Thus, the next step towards implementation would be the development of a memory efficient solver for large-scale unstructured Sylvester equations.

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