Equivalence of Riccati-based Robust Controller Design for Index-1 Descriptor Systems and Standard Plants with Feedthrough

Peter Benner and Jan Heiland

Abstract— **The Riccati-based approach to robust control is completely understood in theory since long but seldom used for large-scale systems in practice. Only recently, we have transferred iterative algorithms that allow the computation of solutions to LQR Riccati equations in the large-scale setting to the indefinite Riccati equations that appear in robust control [2]. For descriptor systems, the relevant Riccati equations are nonsymmetric and, for large-scale systems, there is no established algorithm that can handle these even in the LQR case. In this paper, we show how the general theory for descriptor systems with index-1 pencil coincides with the theory for standard linear time invariant case with feedthrough terms. This provides an algorithm to characterize and to compute the solution to the generalized indefinite Riccati equations via standard Riccati equations. In view of feasibility for large-scale descriptor systems, we illustrate how to arrive at the equivalent standard Riccati-equation without resorting to the canonical form used in the derivation of the theoretical results.**

I. Introduction

We consider a standard class of robust controllers which are dynamical output-based feedback controllers that stabilize a linear-time invariant (LTI) plant while minimizing the gain of disturbance inputs to unmeasured outputs; see [18] for a thorough discussion.

Now consider descriptor systems with pencils of index 1. We recall the commonly known equivalence of such systems with standard LTI systems with feedthrough terms; see, e.g, [7]. Thus, in the case of an index-1 pencil, one may well expect that the general Riccati-based theory for the design of robust controllers for descriptor systems [17] coincides with the theory for standard LTI systems with feedthrough terms [18, Sec. 17].

In fact, once the equivalence of the underlying assumptions is established, the equivalence of solvability of the relevant Riccati equations follows by transitivity of the results on the existence of robust controllers:

Proposition 1: Under standard assumptions, the following statements are equivalent.

(a) The Riccati equations associated with a standard LTI system with feedthrough have stabilizing solutions.

(b) There exists a stabilizing controller with a given robustness margin.

(c) The Riccati equations associated with the equivalent descriptor system have stabilizing solutions.

P. Benner and J. Heiland are with the Max Planck Institute for Dynamics of Complex Technical Systems, 39106 Magdeburg, Germany, {benner,heiland}@mpi-magdeburg.mpg.de In the time when the manuscript was prepared, J. Heiland was with the School of Mechatronic Engineering and Automation, Shanghai University, Shanghai, 200444, China funded by the SHU's Overseas Expert Recruitment Program and 111 Project (No. D18003) of State Administration of Foreign Expert Program.

□

The equivalence of (a) and (b) is shown, e.g., in [18] and that of (b) and (c) in [17]. In this paper, we prove the equivalence of the assumptions used in [17] and in [18] and establish the equivalence of (a) and (c), namely the equivalence of solvability of Riccati equations, directly. This does not advance the theory of robust controller design but comes with the following practical impacts:

- An approach to compute solutions to descriptor-system related Riccati equations via solving standard Riccati equations and vice versa. For standard equations one may apply standard methods. However, the descriptor Riccati equations preserve sparsity of the coefficients, which might be advantageous for large-scale systems.
- An a-priori check for solvability, namely the gain of the feedthrough. This condition is a necessary condition for the existence of a stabilizing controller and thus necessary for solvability of the Riccati equations while not obvious in the case of the descriptor formulation.

This paper focusses on the technical detail of solving Riccati equations for robust controller design. For general considerations of linear quadratic control of descriptor and singular systems see, e.g., [5], [9], [11], [15].

II. Canonical Plant Description and Assumptions

We consider linear time-invariant systems:

$$E\dot{x} = Ax \;\; + B_1 w \;\; + B_2 u, \tag{1a}$$

$$z = C_1 x + D_{11} w + D_{12} u, \tag{1b}$$

$$y = C_2 x + D_{21} w + D_{22} u, \tag{1c}$$

with $B_1 \in \mathbb{R}^{p_1,n}$, $B_2 \in \mathbb{R}^{p_2,n}$, $C_1 \in \mathbb{R}^{q_1,n}$, and $C_2 \in \mathbb{R}^{q_2,n}$ and with $E \in \mathbb{R}^{n,n}$ singular but (E, A) being regular and of index 1. With (E, A) regular and of index 1, without loss of generality, we can assume that (E, A) in (1) is of the form

$$(E, A) = (\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}), \tag{2}$$

with J being square and in Jordan canonical form. Moreover, we will assume

$$B_1 = \begin{bmatrix} B_1^P \\ B_1^I \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_2^P \\ B_2^I \end{bmatrix} \tag{3}$$

and

$$C_1 = \begin{bmatrix} C_1^P \\ C_1^I \end{bmatrix}, \quad C_1 = \begin{bmatrix} C_1^P \\ C_1^I \end{bmatrix} \tag{4}$$

to be partitioned in accordance with (E, A) in (2).

Proposition 2: Let (E, A) as in (2) and B_1, B_2 and C_1, C_2 as in (3) and (4). Then the following systems are equivalent:

1) System (1) written as

$$\left[\begin{array}{c|cc} \begin{bmatrix} -sI+J & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} B_1^P \\ B_1^I \end{bmatrix} & \begin{bmatrix} B_2^P \\ B_2^I \end{bmatrix} \\ \hline \begin{bmatrix} C_1^P & C_1^I \\ C_2^P & C_2^I \end{bmatrix} & \begin{matrix} D_{11} \\ D_{21} \end{matrix} & \begin{matrix} D_{12} \\ D_{22} \end{matrix} \end{array}\right]. \tag{5}$$

2) The standard state space system

$$\left[\begin{array}{c|cc} -sI+J & B_1^P & B_2^P \\ \hline C_1^P & \tilde{D}_{11} & \tilde{D}_{12} \\ C_2^P & \tilde{D}_{21} & \tilde{D}_{22} \end{array}\right], \tag{6}$$

where

$$\begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} - \begin{bmatrix} C_1^I \\ C_2^I \end{bmatrix} \begin{bmatrix} B_1^I & B_2^I \end{bmatrix}.$$

3) The descriptor system with $D = 0$:

$$\left[\begin{array}{c|cc} \begin{bmatrix} -sI+J & \tilde{0} \\ 0 & \tilde{I} \end{bmatrix} & \begin{bmatrix} B_1^P \\ \tilde{B}_1^I \end{bmatrix} & \begin{bmatrix} B_2^P \\ \tilde{B}_2^I \end{bmatrix} \\ \hline \begin{bmatrix} C_1^P & \tilde{C}_1^I \\ C_2^P & \tilde{C}_2^I \end{bmatrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \end{array}\right]. \tag{7}$$

where $\tilde{B}_1^I$, $\tilde{B}_2^I$, $\tilde{C}_1^I$, $\tilde{C}_2^I$ are defined such that

$$\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} - \begin{bmatrix} C_1^I \\ C_2^I \end{bmatrix} \begin{bmatrix} B_1^I & B_2^I \end{bmatrix} = - \begin{bmatrix} \tilde{C}_1^I \\ \tilde{C}_2^I \end{bmatrix} \begin{bmatrix} \tilde{B}_1^I & \tilde{B}_2^I \end{bmatrix}.$$

□

Remark 3: The preceding considerations illustrate how a descriptor system with index-1 pencil can be equivalently realized as a standard LTI system with feedthrough terms and also how a standard LTI system can be realized as a descriptor system with $D = 0$. Accordingly, any descriptor system with a regular pencil (not necessarily of index 1) can be realized with $D = 0$.

The following analysis considers systems with a realization with $D = 0$, $B_2^I = 0$, and $C_2^I = 0$. As summarized in Remark 3, the assumption $D = 0$ is not a restriction. The other assumptions are made to simplify the formulas.

If we extend B_1 and C_1 by zeros, we obtain the equivalent *standard* system

$$\left[\begin{array}{c|cc} \begin{bmatrix} -sI+J & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} B_1^P & 0 \\ B_1^I & 0 \end{bmatrix} & \begin{bmatrix} B_2^P \\ 0 \end{bmatrix} \\ \hline \begin{bmatrix} C_1^P & C_1^I \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} C_2^P & 0 \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right]. \tag{8}$$

By Proposition 2, System (8) is equivalent to the standard LTI system

$$\left[\begin{array}{c|cc} -sI+J & \begin{bmatrix} B_1^P & 0 \end{bmatrix} & \begin{bmatrix} B_2^P \end{bmatrix} \\ \hline \begin{bmatrix} C_1^P \\ 0 \end{bmatrix} & \begin{bmatrix} -C_1^I B_1^I & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} \\ \begin{bmatrix} C_2^P \end{bmatrix} & \begin{bmatrix} 0 & I \end{bmatrix} & 0 \end{array}\right]. \tag{9}$$

III. Notions and Notations

We have already used the notions of *regularity* and *index* of a pair (E, A) of square matrices. Such a pair is called *regular*, if $sE - A$ is invertible for some $s \in \mathbb{C}$. If the pair (E, A) is regular, it is equivalent to a matrix pair

$$\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \tag{10}$$

where N and J are in Jordan canonical form with all eigenvalues of N being zero. The smallest integer ν, such that $N^{\nu-1} \neq 0$ and $N^{\nu} = 0$ is called the (*Kronecker*) *index* of (E, A). With the convention for the zero matrix that $0^0 = I$, the pair in Equation (2) is of index 1.

Let B be an input matrix and consider the system (E, A, B) in its equivalent form

$$\dot{x}^P = Jx^P + B^P u, \tag{11a}$$

$$N\dot{x}^I = x^I + B^I u, \tag{11b}$$

with B partitioned in accordance to (10).

Definition 4: A descriptor system (E, A, B), with (E, A) regular and of index ν is called

- *finite dynamics stable*, if its slow subsystem (11a) is stable;
- *finite dynamics stabilizable*, if its slow subsystem (11a) is stabilizable;
- *impulse controllable* (cp. [6]), if for any initial condition x_0^I for the fast subsystem (11b) and for any vector w and any $\tau \in \mathbb{R}$, there exists an admissible input u from the set of piecewise $\nu-1$-times differentiable functions on $\mathbb{R}$ such that

$$x^I[\tau] = \sum_{i=1}^{\nu-1} \delta_\tau^{(i-1)} N^i w,$$

where $\delta_\tau^{(k)}$ is the k-th distributional derivative of the delta distribution centered at τ, and where $x^I[\tau]$ is the unique distribution z with point support in τ that satisfies

$$N\dot{z} = z - \delta_\tau N(x^I(\tau+) - x^I(\tau-)).$$

□

Impulse controllability means the capability of choosing an admissible control that cancels all distributional parts of the solution that may occur because of inconsistent initial values or disturbance inputs. There are several equivalent algebraic characterizations of impulse controllability; see, e.g., [6], [7].

Since index-1 systems do not develop impulses, they are always impulse controllable. This can also be seen from the canonical form (11) and the equivalent condition (see [7, Thm. 2-2.3]) that

$$\operatorname{image} N = \operatorname{image} \begin{bmatrix} NB^I & N^2B^I & \ldots & N^{\nu-1}B^I \end{bmatrix}$$

that is trivially fulfilled for $N = 0$.

As for standard systems, the concepts of *(finite dynamics) detectability* and *(impulse) observability* of a system (E, A)

with output matrix C can be defined by duality, i.e., by *finite dynamics stabilizability* and *impulse controllability* of the dual system (E^T, A^T, C^T).

Furthermore, we use $\rho(M)$ to denote the spectral radius of a square matrix M (i.e., the largest in modulus eigenvalue of M) and $\bar{\sigma}(M)$ to denote the largest singular value of a, possibly nonsquare, matrix. Also, we write $M > 0$ for M being symmetric positive definite.

For further reference, we introduce the $\mathcal{H}_\infty$-Riccati equations for descriptor systems:

$$A^T\mathcal{X} + \mathcal{X}^TA + C_1^TC_1 + \mathcal{X}^T(\frac{1}{\gamma^2}B_1B_1^T - B_2B_2^T)\mathcal{X} = 0, \tag{12a}$$

$$E^T\mathcal{X} - \mathcal{X}^TE = 0, \tag{12b}$$

and

$$A\mathcal{Y} + \mathcal{Y}^TA^T + B_1^TB_1 + \mathcal{Y}^T(\frac{1}{\gamma^2}C_1C_1^T - C_2C_2^T)\mathcal{Y} = 0, \tag{13a}$$

$$E\mathcal{Y} - \mathcal{Y}^TE^T = 0. \tag{13b}$$

For the presented index-1 case, we have that (E, A) are as in (2) and we assume that B_1, C_1, B_2, and C_2 are as in (8).

Remark 5: A solution $\mathcal{X}$ to the Riccati equation (12) is called *admissible (stabilizing)* if the closed loop system with coefficients $(E, A + (\frac{1}{\gamma^2}B_1^TB_1 - B_2B_2^T)\mathcal{X})$ is of index 1 (and *finite dynamics stable*).

IV. Equivalence of Basic Assumptions

For standard LTI systems, that is (1) with $E = I$, the following assumptions are commonly made

i) (A, B_2) is stabilizable and (C_2, A) is detectable,

ii) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$,

iii) $\begin{bmatrix} -i\omega I + A & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$,

iv) $\begin{bmatrix} -i\omega I + A & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

Here, we used the formulation as in [18, Sec. 17.1] and we will refer to the above assumptions as (ZDG-i)–(ZDG-iv). Note that exactly the same assumptions were used in, e.g., [10], [8]. Except of (ZDG-II), all assumptions are either necessary for the existence of proper stabilizing controllers or made only to simplify the formulas but do not restrict the class of systems under consideration. Assumption (ZDG-II) can be relaxed to

ii*) the matrices D_{12} and D_{21} are of full rank,

whereas a further relaxation could lead to improper controllers; see the discussion in [18, Sec. 17.3].

For the theory for general descriptor systems, the basic assumptions were stated in [17, (A1)–(A7)] or [16]:

i) (E, A) is regular,

ii) (E, A, B_2) is finite dynamics stabilizable and impulse controllable,

iii) (E, A, C_2) is finite dynamics detectable and impulse observable,

iv) $\begin{bmatrix} -i\omega E + A & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$ and is row reduced,

v) $\begin{bmatrix} -i\omega E + A & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$ and is column reduced,

vi) $D_{12}^TD_{12} > 0$,

vii) $D_{21}D_{21}^T > 0$.

We will refer to these assumption as (WYC-i)–(WYC-vii).

For the particular descriptor system (8) and its equivalent formulation as the standard LTI system with feedthrough (9), we show that the preceding assumptions for the descriptor and the standard system are equivalent, too.

Theorem 6: Consider the descriptor system (8) and its equivalent realization (9). Then the assumptions (ZDG-i), (ZDG-ii*), (ZDG-iii), and (ZDG-iv) with respect to (9) hold, if and only, assumptions (WYC-ii)–(WYC-vii) hold with respect to (8).

Proof: In particular, regular index-1 pencils are impulse controllable and observable. If brought into the canonical form (2), the finite dynamics stabilizability/detectability is equivalent to the stabilizability/detectability of (J, B_2^P) and (C_2^P, J). ∎

V. Equivalence of the Riccati equations

To show correspondence of solutions to Riccati equations, we will consider the associated Hamiltonian matrix pencils shown in $(\mathcal{H}_\infty)$ and (H_∞)

$$\begin{bmatrix} -sI + J & 0 & B_2^PB_2^{P^T} - \gamma^{-2}B_1^PB_1^{P^T} & -\gamma^{-2}B_1^PB_1^{I^T} \\ 0 & I & -\gamma^{-2}B_1^IB_1^{P^T} & -\gamma^{-2}B_1^IB_1^{I^T} \\ C_1^{P^T}C_1^P & C_1^{P^T}C_1^I & -sI - J^T & 0 \\ C_1^{I^T}C_1^P & C_1^{I^T}C_1^I & 0 & -I \end{bmatrix}, \tag{$\mathcal{H}_\infty$}$$

$$\begin{bmatrix} -sI + J & 0 \\ C_1^{P^T}C_1^P & -sI - J^T \end{bmatrix} - \begin{bmatrix} -B_1^P(-\gamma^2 + B_1^{I^T}C_1^{I^T}C_1^IB_1^I)^{-1}B_1^{I^T}C_1^{I^T}C_1^P & B_2^PB_2^{P^T} + B_1^P(-\gamma^2 + B_1^{I^T}C_1^{I^T}C_1^IB_1^I)^{-1}B_1^{P^T} \\ -C_1^{P^T}C_1^IB_1^I(-\gamma^2 + B_1^{I^T}C_1^{I^T}C_1^IB_1^I)^{-1}B_1^{I^T}C_1^{I^T}C_1^P & C_1^{P^T}C_1^IB_1^I(-\gamma^2 + B_1^{I^T}C_1^{I^T}C_1^IB_1^I)^{-1}B_1^{P^T} \end{bmatrix} \tag{H_∞}$$

as well as those shown in $(\mathcal{J}_\infty)$ and (J_∞)

$$\begin{bmatrix} -sI+J^T & 0 & C_2^P {C_2^P}^T - \gamma^{-2} {C_1^P}^T C_1^P & -\gamma^{-2} {C_1^I}^T C_1^P \\ 0 & I & -\gamma^{-2} {C_1^P}^T C_1^I & -\gamma^{-2} {C_1^I}^T C_1^I \\ B_1^P {B_1^P}^T & B_1^I {B_1^P}^T & -sI-J & 0 \\ B_1^P {B_1^I}^T & B_1^I {B_1^I}^T & 0 & -I \end{bmatrix}, \qquad (\mathcal{J}_\infty)$$

$$\begin{bmatrix} -sI+J^T & 0 \\ B_1^P {B_1^P}^T & -sI-J \end{bmatrix}$$
$$- \begin{bmatrix} -{C_1^P}^T(-\gamma^2 + C_1^I B_1^I {B_1^I}^T {C_1^I}^T)^{-1} C_1^I B_1^P {B_1^I}^T & {C_2^P}^T C_2^P + {C_1^P}^T(-\gamma^2 + C_1^I B_1^I {B_1^I}^T {C_1^I}^T)^{-1} C_1^P \\ -B_1^I {B_1^P}^T {C_1^I}^T(-\gamma^2 + C_1^I B_1^I {B_1^I}^T {C_1^I}^T)^{-1} C_1^I B_1^P {B_1^I}^T & B_1^I {B_1^P}^T {C_1^P}^T(-\gamma^2 + C_1^I B_1^I {B_1^I}^T {C_1^I}^T)^{-1} C_1^I \end{bmatrix} \qquad (J_\infty)$$

that are associated with systems (9) and (8). In fact, the existence of stabilizing solutions to the Riccati equation (12) is completely characterized by the existence of invariant subspaces to $(\mathcal{H}_\infty)$; see [12, Lem. 1], just like the pencil (H_∞) characterizes the Riccati feedback matrices that are relevant for the standard LTI case (9); see [18, Thm. 17.1]

Theorem 7: Consider systems (8) and (9) and let Assumptions (ZDG-i)–(ZDG-iv) or, equivalently, Assumptions (WYC-i)–(WYC-vii) hold. Then the following two statements are equivalent.

(C1) The Riccati equations associated with the Hamiltonian pencils H_∞, J_∞ possess stabilizing solutions X_∞, $Y_\infty \geq 0$ with $\rho(X_\infty Y_\infty) < \gamma^2$ and $\bar\sigma(C_1^I B_1^I) < \gamma$.

(C2) The Riccati equations associated with the Hamiltonian pencils $\mathcal{H}_\infty$, $\mathcal{J}_\infty$ possess stabilizing solutions $\mathcal{X}_\infty$, $\mathcal{Y}_\infty$ with $\rho(\mathcal{X}_\infty \mathcal{Y}_\infty) < \gamma^2$.

□

Remark 8: As mentioned above, in the presented setup, condition (C1) is necessary and sufficient for the existence of γ-stabilizing controllers for (9), whereas the condition (C2) is sufficient for the existence of γ-stabilizing controllers for (8). Since the systems are equivalent (see Remark 3) and since the underlying assumptions are the same (see Theorem 6) the claim of the preceding theorem follows immediately by transitivity of implications. Here, however, we provide a direct proof via the correspondence of the Riccati solutions.

In order to not interrupt the argument in the proof of Theorem 7, we state two auxiliary results first.

Lemma 9: Let $\gamma > 0$ and let $B_1^I \in \mathbb{R}^{k\times m}$ and $C_1^I \in \mathbb{R}^{l\times k}$ for some k, l, $m \in \mathbb{N}$. Then the following statements are equivalent:

1.)
$$X + X^T + {C_1^I}^T C_1^I + \frac{1}{\gamma^2} X^T B_1^I {B_1^I}^T X = 0$$
has a unique symmetric (negative) definite solution such that $-I - \frac{1}{\gamma^2} B_1^I {B_1^I}^T X$ is stable.

2.) $\bar\sigma(C_1^I B_1^I) < \gamma$.

Proof: We show that $\bar\sigma(C_1^I B_1^I) < \gamma$ is equivalent to the system

$$\dot{x} = -x + B_1^I u,$$
$$y = C_1^I x,$$

having an $\mathcal{L}_2$ gain, i.e. the largest singular value of the transfer function evaluated on the imaginary axis, smaller than γ, so that the claim follows by [13, Thm. 2.3.1]. In fact, the transfer function is given as $G(s) = C_1^I(sI+I)^{-1}B_1^I$ from which it follows that

$$\max_{\omega\in\mathbb{R}} \bar\sigma(G(i\omega)) = \max_{\omega\in\mathbb{R}} \frac{1}{(1-i\omega)(1+i\omega)} \bar\sigma(C_1^I B_1^I) = \frac{1}{1+\omega^2}\bar\sigma(C_1^I B_1^I)$$

which is less than γ if and only if $\bar\sigma(C_1^I B_1^I)$ is less than γ. ■

Lemma 10: If the Riccati equation (12a) with coefficients as in (8) has a stabilizing solution then, $\bar\sigma(C_1^I B_1^I) < \gamma$.

Proof: Let $\mathcal{X}_\infty$ be the stabilizing solution to (12). Then $\tilde{\mathcal{X}}_\infty := \mathcal{X}_\infty$ is a stabilizing solution to

$$(A^T - \mathcal{X}_\infty^T B_2 B_2^T)\mathcal{X} + \mathcal{X}^T(A - B_2 B_2^T \mathcal{X}_\infty) + C_1^T C_1 + \mathcal{X}_\infty^T B_2 B_2^T \mathcal{X}_\infty + \frac{1}{\gamma^2}\mathcal{X}^T B_1 B_1^T \mathcal{X} = 0,$$

which is a refactoring of Equation (12). By the *bounded real lemma* for descriptor systems, see [17, Lem. 5], it follows that

$$\gamma^2 I - \tilde{B}^I {\tilde{C}^I}^T \tilde{C}^I {\tilde{B}^I}^T > 0 \qquad (14)$$

where $\tilde{B}^I$, $\tilde{C}^I$ correspond to partitioning of the canonical form (2) of $(E, A - B_2 B_2^T \mathcal{X}_\infty)$ applied the input matrix $\tilde{B} = B_1$ and $\tilde{C} = \begin{bmatrix} C_1 & B_2^T \mathcal{X}_\infty \end{bmatrix}$. With (E, A) already in the canonical form, by the triangular structure of $\mathcal{X}_\infty$, and since $B_2^P = 0$, it follows that the transformation to the canonical form only affects the dynamical part and that $\tilde{B}^I {\tilde{C}^I}^T = B_1^I {C_1^I}^T$ so that (14) implies $\bar\sigma(C_1 B_1) < \gamma$. ■

We can now provide the proof of the main theorem.

Proof: (of Theorem 7)

We first show how H_∞ and $\mathcal{H}_\infty$ correspond to each other. Then, the equivalence of the associated Riccati solutions X_∞ and $\mathcal{X}_\infty$ can be established right away. The case of J_∞ and $\mathcal{J}_\infty$ and, thus, Y_∞ and $\mathcal{Y}_\infty$, works analogously.

From existence and admissibility of $\mathcal{X}_\infty$, we can infer that $\rho(C_2^I B_2^I) < \gamma$; see Lemma 10. Thus, the invertibility of

$$I - \gamma^{-2} B_1^I {B_1^I}^T {C_1^I}^T C_1^I$$

as it used for the following transformations is secured either immediately by assumption (C1) or as an implication of (C2).

By swapping the second and third columns and rows – which is a congruence transformation that preserves the Hamiltonian structure – we find that $\mathcal{H}_\infty$ is equivalent to

$$\begin{bmatrix} -sI + J & B_2^P {B_2^P}^T - \gamma^{-2} B_1^P {B_1^P}^T & 0 & -\gamma^{-2} B_1^P {B_1^I}^T \\ {C_1^P}^T C_1^P & -sI - J^T & {C_1^P}^T C_1^I & 0 \\ 0 & -\gamma^{-2} B_1^I {B_1^P}^T & I & -\gamma^{-2} B_1^I {B_1^I}^T \\ {C_1^I}^T C_1^P & 0 & {C_1^I}^T C_1^I & -I \end{bmatrix}$$

which we rewrite with 2×2 blocks like

$$\mathcal{H}_\infty \sim \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_3 & \mathcal{T}_4 \end{bmatrix},$$

see also [17, p. 317]. Expressing the inverse of

$$\mathcal{T}_4 = \begin{bmatrix} I & -\gamma^{-2} B_1^I {B_1^I}^T \\ {C_1^I}^T C_1^I & -I \end{bmatrix}$$

as

$$\begin{bmatrix} I & 0 \\ {C_1^I}^T C_1^I & I \end{bmatrix} \begin{bmatrix} (I - \gamma^{-2} B_1^I {B_1^I}^T {C_1^I}^T C_1^I)^{-1} & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & -\gamma^{-2} B_1^I {B_1^I}^T \\ 0 & I \end{bmatrix}$$

we infer that $\mathcal{T}_4^{-1}$ exists so that invariant subspaces to $\mathcal{H}_\infty$ can be put into one-to-one correspondence with invariant subspaces to the Hamiltonian pencil

$$\mathcal{T}_1 - \mathcal{T}_3 \mathcal{T}_4^{-1} \mathcal{T}_2,$$

which, as can be confirmed by straight-forward computations, exactly coincides with H_∞. Accordingly, a Riccati solution to H_∞ defines invariant subspaces to $\mathcal{H}_\infty$. It remains to show that and how stabilizing solutions correspond to each other.

Let (C1) hold. Then, as proposed in the proof of [17, Prop. 1], set

$$\mathcal{X} := \begin{bmatrix} X_\infty & 0 \\ X_{12} & X_{22} \end{bmatrix},$$

where

$$X_{12} := -\begin{bmatrix} -X_{22} & I \end{bmatrix} \mathcal{T}_4^{-1} \mathcal{T}_3 \begin{bmatrix} I \\ X_\infty \end{bmatrix}$$

and where X_{22} is the unique symmetric negative definite solution to the T-Riccati equation

$$ {C_1^I}^T C_1^I + \frac{1}{\gamma^2} X_{22}^T B_1^I {B_1^I}^T X_{22} + X_{22} + X_{22}^T = 0 \qquad (15)$$

which, by virtue of (C1) and Lemma 9, exists. With that construction one can confirm that $\mathcal{X}$ provides a solution to the Riccati equation associated with $\mathcal{H}_\infty$. Since $X_\infty = X_\infty^T \geq 0$ and since, if in this triangular form, $\mathcal{X}$ is stabilizing, if and only if the upper left block is symmetric positive definite (as follows from an extension of [14, Lem. 3.4] to the indefinite case with the arguments of [18, Lem. 16.6]), we can conclude that $\mathcal{X}$ is a stabilizing Riccati solution associated with $\mathcal{H}_\infty$. By analogous considerations for the correspondence of $\mathcal{J}_\infty$ and J_∞, one finds that (C1) implies that

$$\mathcal{Y} := \begin{bmatrix} Y_\infty & 0 \\ Y_{12} & Y_{22} \end{bmatrix}$$

is a stabilizing Riccati solution associated with $\mathcal{J}_\infty$ with $Y_\infty \geq 0$ and $-Y_{22} \geq 0$. It remains to prove the spectrum condition $\rho(\mathcal{X}_\infty \mathcal{Y}_\infty) < \gamma^2$. This follows from the existence of the admissible solution

$$\mathcal{Z}_\infty = \begin{bmatrix} Y_{tmp} & 0 \\ Z_{12} & Z_{22} \end{bmatrix}$$

to the auxiliary Riccati equation

$$\begin{aligned} (A + \frac{1}{\gamma^2} B_1 B_1^T \mathcal{X}_\infty)\mathcal{Z} + \mathcal{Z}^T (A^T + \frac{1}{\gamma^2} \mathcal{X}_\infty B_1 B_1^T)\mathcal{Z} \\ B_1 B_1^T + \mathcal{Z}^T (\frac{1}{\gamma^2} \mathcal{X}_\infty B_2 B_2^T \mathcal{X}_\infty - C_2^T C_2)\mathcal{Z} = 0, \\ E\mathcal{Z} = \mathcal{Z}^T E^T \end{aligned}$$

that exists by virtue of the correspondence of the associated Hamiltonian pencil with the Hamiltonian pencil that defines Y_{tmp} as used in [18, Sec. 16.8.2], and by a standard argument for the *Lyapunov* equation for Z_{22} that results from the particular structure of the coefficients. Once the existence of the stabilizing $\mathcal{Z}_\infty$ is established, the spectrum condition follows from [17, Cor. 8].

For the reverse direction, let (C2) hold. Then $\mathcal{X}_\infty$ has the form

$$\mathcal{X}_\infty := \begin{bmatrix} X_{11} & 0 \\ X_{12} & X_{22} \end{bmatrix},$$

as can be concluded from the necessary condition that $E^T \mathcal{X}_\infty = \mathcal{X}_\infty^T E$ that is encoded in the condition that $\begin{bmatrix} \mathcal{X}_\infty \\ I \end{bmatrix}$ has to define an invariant subspace for $\mathcal{H}_\infty$ for all $s \in \mathbb{C}$. This also means that $X_{11}^T = X_{11}$, and, as it again follows with an extension of [14, Lem. 3.4] to the indefinite case, it holds that $X_{11} \geq 0$. By the arguments above that put $\mathcal{H}_\infty$ and H_∞ in correspondence it follows that X_{11} is a solution to the H_∞-related Riccati equation and because of its positive definiteness it is stabilizing. Analogous arguments provide that

$$\mathcal{Y}_\infty = \begin{bmatrix} Y_{11} & 0 \\ Y_{12} & Y_{22} \end{bmatrix},$$

with Y_{11} defining a stabilizing solution to the Riccati equation associated with J_∞.

Moreover, because of the triangular structure of $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$, the spectral radius of $X_\infty Y_\infty$ can only be smaller than that of $\mathcal{X}_\infty \mathcal{Y}_\infty$ which is smaller than γ^2.

Finally, as worked out in the proof of Lemma 10, from $\mathcal{X}_\infty$ being a stabilizing solution to (12) and from the structure of the coefficients, it follows that $\bar{\sigma}(C_1^I B_1^I) < \gamma$. ■

VI. Practical Aspects

A common case in applications [3] would be that a descriptor system with a regular index-1 pencil (E, A) is given as

$$\left(\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1^d \\ B_1^a \end{bmatrix}, \begin{bmatrix} B_2^d \\ B_2^a \end{bmatrix}, \begin{bmatrix} C_1^d \\ C_1^a \end{bmatrix}, \begin{bmatrix} C_2^d \\ C_2^a \end{bmatrix}, 0 \right) \qquad (16)$$

with A_{22} being invertible and an invertible matrix M like mass matrix from a finite element discretization.

Such a system can be brought into the form

$$\left(\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} - A_{12} A_{22}^{-1} A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}, \dots, 0 \right) \qquad (17)$$

without changing the blocks of B_1^a, B_2^a, C_1^a, and C_2^a.

Since the particular structure of J was of no importance in the formulation of Theorem 7, it can be (formally) replaced by $A_{11} - A_{12}A_{22}^{-1}A_{21}$ and the equivalence results can be applied right away if

1) if $B_2^a = 0$ and $C_2^a = 0$,
2) B_1^I is (formally) replaced by $A_{22}^{-1}B_1^a$, and
3) the matrix sI is replaced by sM or sM^T.

Remark 11: As for the inverse of A_{22} appearing in 2.), one may consider not computing it explicitly but only applied to vectors, e.g., if iterative procedures are used to compute solutions to the Riccati equations. Also, the number of algebraic conditions might be small in comparison to the dynamical equations so that $A_{11} - A_{12}A_{22}^{-1}A_{21}$ can be treated as a low-rank update to A_{11}.

As for 3.), the appearance of M in the formulas. Equivalence of the formulas follows immediately through a suitable scaling by M^{-1} or M^{-T}. For the practical realization, however, one may call on the generalized Riccati equations that avoid the inversion of M, see, e.g, [1] for the formulas for LQG controllers.

Then the presented equivalences of the conditions and the Riccati equations can be exploited for numerical considerations in the following ways.

1) The computation of the largest singular value of $C_1^a A_{22}^{-1} B_1^a$ immediately gives a lower bound on γ for which the $\mathcal{H}_\infty$, $\mathcal{J}_\infty$ may define Riccati equations with stabilizing solutions.
2) If a solver for standard Riccati equations is available, together with the considerations laid out in Remark 11, the construction in the proof of Theorem 7 can be used to compute Riccati solution for $\mathcal{H}_\infty$, $\mathcal{J}_\infty$ in terms of Riccati solutions associated with H_∞, J_∞.
3) If a system of the form (9) with a nonzero D_{11} is given, the matrices that make up the Hamiltonian pencils H_∞, J_∞ and define the coefficients in the Riccati equations contain inverses and products of the feedthrough matrices; compare, e.g., the left upper block in H_∞ that reads
$$J + B_1^P(-\gamma^2 + {B_1^I}^T {C_1^I}^T C_1^I B_1^I)^{-1} {B_1^I}^T {C_1^I}^T C_1^P.$$
In particular in large scale settings, it, thus, might be beneficial to factorize the D_{11} matrix and extend the system as described in Remark 3 into a descriptor system. Then the associated Hamiltonian pencils $\mathcal{H}_\infty$, $\mathcal{J}_\infty$ can be considered.

VII. Conclusion

We have established a direct correspondence between the Riccati solution associated with Hamiltonian pencils for index-1 descriptor systems and their equivalent formulation as a standard LTI. It is well-known that these Riccati solutions can be used to design robust feedback controllers.

We have also discussed, how this correspondence could be exploited for numerical methods to compute Riccati solutions. The development of such methods in general, however, is open for the future. Especially, up to now, there are only solvers for the symmetric Riccati equations associated with descriptor systems [4] but not for the relevant unsymmetric equations let alone the considered indefinite case.

Another possible extension would be the case of higher index pencils. Depending on B and C, a higher index does not necessarily mean that the system is improper, so that equivalence to a standard LTI problem can still be guaranteed. If the input *is* differentiated, then, because of the generally assumed impulse controllability, the resulting closed loop system will be proper, i.e. in the form considered here.

References

[1] P. Benner and J. Heiland. LQG-Balanced Truncation low-order controller for stabilization of laminar flows. In R. King, editor, *Active Flow and Combustion Control 2014*, volume 127 of *Notes on Numerical Fluid Mechanics and Multidisciplinary Design*, pages 365–379. Springer, Berlin, 2015.

[2] P. Benner, J. Heiland, and S. W. Werner. Robust controller versus numerical model uncertainties for stabilization of Navier-Stokes equations. *IFAC-PapersOnLine*, 52(2):25 – 29, 2019. 3rd IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2019.

[3] P. Benner, J. Saak, and M. M. Uddin. Balancing based model reduction for structured index-2 unstable descriptor systems with application to flow control. *Numer. Algebra Control Optim.*, 6(1):1–20, Mar. 2016.

[4] P. Benner and T. Stykel. Numerical solution of projected algebraic Riccati equations. *SIAM J. Numer. Anal.*, 52(2):581–600, 2014.

[5] C. Bhawal and D. Pal. Almost every single-input LQR optimal control problem admits a PD feedback solution. *IEEE Control Systems Letters*, 3(2):452–457, 2019.

[6] D. Cobb. Controllability, observability, and duality in singular systems. *IEEE Trans. Autom. Control*, 29(12):1076–1082, 1984.

[7] L. Dai. *Singular Control Systems*. Number 118 in Lecture Notes in Control and Information Sciences. Springer, Berlin, 1989.

[8] J. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Automat. Control*, 34:831–847, 1989.

[9] A. Ferrante and L. Ntogramatzidis. Continuous-time singular linear-quadratic control: Necessary and sufficient conditions for the existence of regular solutions. *Sys. Control Lett.*, 93:30–34, 2016.

[10] K. Glover and J. C. Doyle. State-space formulae for all stabilizing controllers that satisfy an H_∞-norm bound and relations to risk sensitivity. *Syst. Control Lett.*, 11:167–172, 1988.

[11] R. K. Kalaimani, M. N. Belur, and D. Chakraborty. Singular LQ control, optimal PD controller and inadmissible initial conditions. *IEEE Trans. Automat. Contr.*, 58(10):2603–2608, 2013.

[12] A. Kawamoto, K. Takaba, and T. Katayama. On the generalized algebraic Riccati equation for continuous-time descriptor systems. *Linear Algebra Appl.*, 296(1-3):1–14, 1999.

[13] H. W. Knobloch, A. Isidori, and D. Flockerzi. *Topics in Control Theory*, volume 22. Basel: Birkhäuser Verlag, 1993.

[14] J. Möckel, T. Reis, and T. Stykel. Linear-quadratic Gaussian balancing for model reduction of differential-algebraic systems. *Internat. J. Control*, 84(10):1627–1643, 2011.

[15] T. Reis, O. Rendel, and M. Voigt. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra Appl.*, 485:153–193, 2015.

[16] K. Takaba, N. Morihira, and T. Katayama. H_∞ control for descriptor systems – a J-spectral factorization approach –. In *Proceedings of the 33rd IEEE Conference on Decision and Control (CDC)*, volume 3, pages 2251–2256, 1994.

[17] H.-S. Wang, C.-F. Yung, and F.-R. Chang. Bounded real lemma and H_∞ control for descriptor systems. *IEE Proceedings - Control Theory and Applications*, 145(3):316–322, 1998.

[18] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Upper Saddle River, NJ, 1996.